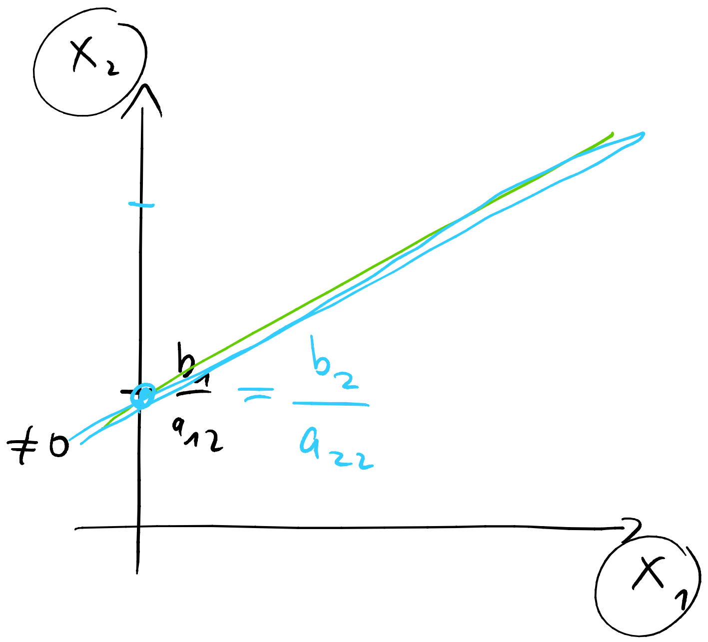
13.09.18 - LGS

Donnerstag, 13. September 2018 11:23



$$\boxed{a_{11}x_1 + a_{12}x_2 = b_1} \quad \Big| -a_{11}x_1$$

$$\Leftrightarrow \quad a_{12}x_2 = b_1 - a_{11}x_1 \quad \Big| : a_{12} \neq 0$$

$$\Leftrightarrow \quad x_2 = -\frac{a_{11}}{a_{12}} x_1 + \frac{b_1}{a_{12}}$$

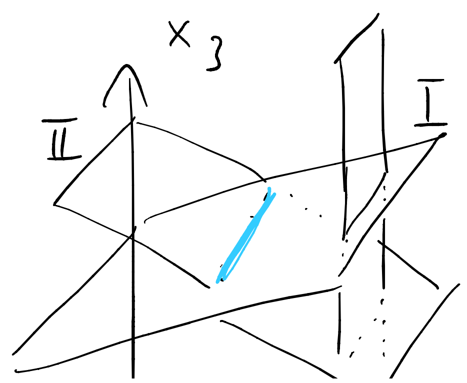
$$y = m x + b$$

$$\boxed{a_{21}x_1 + a_{22}x_2 = b_2}$$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \\ a_{31}x_1 + a_{32}x_2 = b_3 \end{array}\right\} \text{LGS} \mathrel{\hat{=}} 3 \text{ Geraden im } \mathbb{R}^2$$

$$\longrightarrow \quad \begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{array}\right)$$

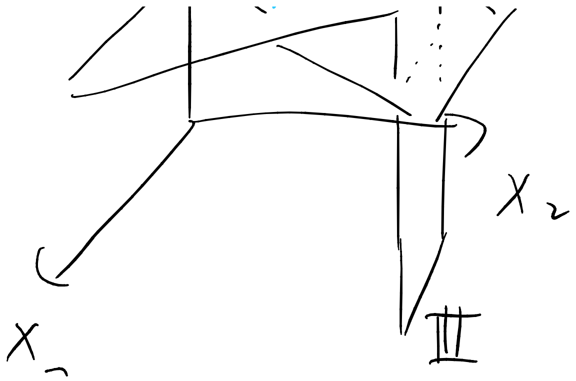
$\mathbb{L}$ = Lösungsmenge = <u>Schnittmenge</u> der drei Geraden



$$\text{I} \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$\text{II} \quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$\begin{array}{l} \text{II} \quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ \text{III} \quad a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}$$



LGS im $\mathbb{R}^3$

$\mathbb{L}$ = Schnittmenge der drei Ebenen I, II, III

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right)$$

a_{12}

Spalte

Zeile

$$\left(\begin{array}{cc|c} 2 & 2 & 4 \\ 2 & -2 & 0 \end{array}\right)$$

$$\begin{array}{ccc} x_1 & x_2 & \\ \multicolumn{3}{c}{\left(\begin{array}{cc|c} 2 & 2 & 4 \\ 0 & 4 & 4 \end{array}\right)} \end{array}$$

$$\begin{array}{ll} \text{I} & 2x_1 + 2x_2 = 4 \\ \text{II} & 2x_1 - 2x_2 = 0 \\ \text{II} & \Leftrightarrow 2x_1 = 2x_2 \end{array}$$

In I einsetzen: $2x_2 + 2x_2 = 4$

$\Leftrightarrow \quad 4x_2 = 4$

$\Leftrightarrow \quad x_2 = 1$

$$4x_2 = 4$$

$$\Leftrightarrow \quad x_2 = 1$$

$x_2 = 1$ in I: $\quad 2x_1 + 2 \cdot \overset{\overset{1}{\shortparallel}}{x_2} = 4$

$$\Leftrightarrow \quad 2x_1 + 2 \cdot 1 = 4$$

$$\Leftrightarrow \quad x_1 = 1$$

Achtung! Hohe Fehlerquote ⟶ Üben!!!

Danach **Probe**!

$$\text{I} \quad 2\underbrace{x_1}_{=1} + 2\underbrace{x_2}_{=1} = 4$$

$$\text{II} \quad 2 \cdot 1 - 2 \cdot 1 = 0 \quad ✓$$

$$\Rightarrow \quad x_1 = 1$$

$$\Rightarrow \quad \mathbb{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$= \{ (1,1) \}$$

$$\left(\begin{array}{cc|c} 2 & 2 & 4 \\ 1 & -1 & 0 \end{array} \right) \Big| \cdot 2$$

$\text{II} \cdot 2$

$$\begin{array}{lrcl} \text{I} & 2x_1 + 2x_2 & = & 4 \\ \text{II} & x_1 - x_2 & = & 0 \quad | \cdot 2 \end{array}$$

I $\quad 2x_1 + 2x_2 = 4$

$$\begin{pmatrix} 2 & 2 & | & 4 \\ 2 & -2 & | & 0 \end{pmatrix}$$

I−II

$$\begin{pmatrix} 2 & 2 & | & 4 \\ 2-2 & 2-(-2) & | & 4-0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & | & 4 \\ 0 & 4 & | & 4 \end{pmatrix}$$

I $\quad 2x_1 + 2x_2 = 4$

II $\quad 2x_1 - 2x_2 = 0$

I $\quad 2x_1 + 2x_2 = 4$

II $\quad (2-2)x_1 + (2-(-2))x_2 = 4-0$

I $\quad 2x_1 + 2x_2 = 4$

II $\quad 4x_2 = 4$

II $\iff x_2 = 1$

In I einsetzen:

$2 \cdot x_1 + 2 \cdot 1 = 4$

$$\Leftrightarrow \quad x_1 = 1$$

$$\mathbb{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1, x_2) = (1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Bsp: $\text{I} \quad 2x_1 - 3x_2 = 3$

$\text{II} \quad \frac{1}{3}x_1 + x_2 = 2$

Es gibt 3 elementare Zeilenumformungen.

1. Vertauschen der Zeilen

$\text{I} \quad \frac{1}{3}x_1 + x_2 = 2$

$\text{II} \quad 2x_1 - 3x_2 = 3$

2. Multiplizieren / Division einer Zeile mit einer Zahl $a \neq 0$

z.B. $3 \cdot \text{II}$

z.B. $3 \cdot \overline{II}$

$$\boxed{\begin{array}{ll} I & \underline{2x_1} - \underline{3x_2} = 3 \\ II & \underline{x_1} + 3x_2 = 6 \end{array}}$$

3. Zeilen addieren/subtrahieren

$$I + II \longrightarrow (2+1)x_1 + \underbrace{(-3+3)}_{=0}x_2 = 3+6$$

$$\Leftrightarrow \quad 3x_1 = 9 \quad \Rightarrow \quad x_1 = 3$$

2+3. = 3' Multipliziere Zeile mit $a \neq 0$ und addiere zur anderen Zeile

$$(-2)\,II + I \longrightarrow \underbrace{(2-2)}_{=0}x_1 + (-3-6)x_2 = 3-12$$

$$\Leftrightarrow \quad -9x_2 = -9$$

$$\Rightarrow \quad x_2 = 1$$

$$\Rightarrow x_2 = 1$$

$$\Rightarrow \mathbb{L} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} = \{ x_1 = 3, \; x_2 = 1 \}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} -3 & -9 & 3 & -12 \\ 3 & 4 & 0 & 11 \\ 2 & 1 & 1 & 7 \\ 2 & -4 & 4 & 6 \end{array} \right) \quad |\cdot(-3) \;]+ \quad |\cdot(-2) \;]+ \quad |\cdot(-2) \;]+$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -1 \\ \cancel{0} & \cancel{-5} & \cancel{3} & \cancel{-1} \\ \cancel{0} & \cancel{-10} & \cancel{6} & \cancel{-2} \end{array} \right) \quad |(-2) \;]+$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 3 & 6 \end{array} \right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -5 & 3 & -1 \end{array} \right)$$

$$3x_3 = 6$$

$$x_3 = 2$$

$$\text{II} \quad -5x_2 + 3x_3 = -1$$

Nach x_3 auflösen: $\boxed{3x_3 = -1 + 5x_2} \iff x_3 = -\frac{1}{3} + \frac{5}{3}x_2$

In $3 \cdot I$ einsetzen:

$$3 \cdot I \qquad 3x_1 + 9x_2 \boxed{-3x_3} = 12$$

$$\Rightarrow \quad 3x_1 + 9x_2 - (-1 + 5x_2) = 12$$

$$\iff \quad 3x_1 + 9x_2 + 1 - 5x_2 = 12$$

$$\iff \quad \underline{3x_1 + 4x_2 = 11}$$

2 Variablen bleiben übrig, d.h. eine Variable kann frei gelassen werden, z.B. $\underline{x_2 = t}$

$$\Rightarrow \quad 3x_3 = -1 + 5t \quad |:3 \qquad \iff \quad x_3 = -\frac{1}{3} + \frac{5}{3}t$$

$$\Rightarrow \quad 3x_1 = 11 - 4t \quad |:3 \qquad \iff \quad x_1 = \frac{11}{3} - \frac{4}{3}t$$

$$\mathbb{L} = \Big\{ \Big(x_1 \quad \Big(\frac{11}{3} - \frac{4}{3}t \quad \Big(\frac{11}{3} \quad \Big(-\frac{4}{3}$$

$$\mathbb{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} - \frac{4}{3}t \\ t \\ -\frac{1}{3} + \frac{5}{3}t \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{11}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix} + t \begin{pmatrix} -\frac{4}{3} \\ 1 \\ \frac{5}{3} \end{pmatrix}}_{\text{Gerade im } \mathbb{R}^3 \text{ in Parameterform}} : t \in \mathbb{R} \right\}$$

4 Ebenen im $\mathbb{R}^3$, die geschnitten werden $\longrightarrow$ LGS

$\mathbb{L}$ = Gerade im $\mathbb{R}^3$.

$$\underbrace{0 \cdot x + 0 \cdot y \cdot 0 \cdot z}_{=0} = 2$$

$$0 = 2$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc = \text{Determinante}$$

$$\| $$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

$$= aei - ahf - bdi + bgf + cdh - cge$$

Laplace-Entwicklung, Regel von Sarrus (nur 3x3)

Wo taucht det in LGS auf???

$$\left(\begin{array}{cc|c} a & b & s \\ c & d & t \end{array}\right) \Big| \cdot \left(-\frac{c}{a}\right) \quad (a \neq 0) \qquad \begin{aligned} a x_1 + b x_2 &= s \\ c x_1 + d x_2 &= t \end{aligned}$$

$$\left(\begin{array}{cc|c} a & b & s \\ 0 & -\frac{bc}{a} + d & -\frac{sc}{a} + t \end{array}\right)$$

$$\left(-\frac{bc}{a} + d\right) x_2 = -\frac{sc}{a} + t \quad \Big| \cdot a$$

$$\Longleftrightarrow \left(-bc + ad\right) \underline{x_2} = -sc + at = \det\begin{pmatrix} a & s \\ c & t \end{pmatrix}$$

$$\overbrace{ad-bc} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

Fall, $\underline{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0}$ $\Rightarrow$ $x_2 = \dfrac{\det\begin{pmatrix} a & s \\ c & t \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$

$$\Rightarrow \underbrace{x_1 = \frac{\det\begin{pmatrix} b & s \\ d & t \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}}_{\text{Cramersche Regel}}$$