Real and complex Brunn-Minkowski theory

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The Brunn Minkowski theorem.

Let A_0 and A_1 be convex bodies in R^n. Denote by $|A|$ the (Lebesgue) volume of $|A|$.

Theorem

$$|A_0 + A_1|^{1/n} \geq |A_0|^{1/n} + |A_1|^{1/n}.$$

We will give a number of 'equivalent' formulations.

Let $A_t := tA_1 + (1-t)A_0$. Then

$$|A_t|^{1/n} \quad \text{is a concave function of } \ t. \tag{1}$$

$$\log|A_t| \quad \text{is a concave function of } \ t. \tag{2}$$

$$|A_t| \geq \min\left(|A_0|, |A_1|\right). \tag{3}$$

B-M implies (1). It is also clear that (1) implies (2] which implies (3). But, they are actually all equivalent.

Proof

It suffices to show that (3) implies B-M.
Let

$$t = \frac{|A_1|^{1/n}}{|A_0|^{1/n} + |A_1|^{1/n}}.$$

Then

$$1 - t = \frac{|A_0|^{1/n}}{|A_0|^{1/n} + |A_1|^{1/n}}.$$

(3) implies that

$$|tA_1/|A_1|^{1/n} + (1-t)A_0/|A_0|^{1/n}| \geq 1.$$

This gives B-M.

An application

Let B be the unit ball and put

$$f(t) := |A + tB|.$$

Then $f'(0) = |\partial A|$. B-M implies that for $t > 0$

$$f^{1/n} \geq |A|^{1/n} + t|B|^{1/n}. \tag{4}$$

Hence

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq n|B|^{1/n}.$$

But equality holds when $A = B$ (!). Hence we get the isoperimetric inequality

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq \frac{|\partial B|}{|B|^{1-1/n}}.$$

Yet another reformulation

Let $\mathcal{A}$ be a convex body in $\mathbb{R}^{n+1}$ and put

$$A_t = \{x \in \mathbb{R}^n; (t, x) \in \mathcal{A}\}.$$

Then

$$\log |A_t|$$

is a concave function of t.

Function version of B-M

Let $\phi(t, x)$ be a convex function on $\mathbb{R}^{n+1}$. Let

$$\tilde{\phi}(t) := -\log \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx.$$

We then have the following generalization of B-M, due to Prékopa:

Theorem

$\tilde{\phi}$ is convex function of t.

The version of B-M on the previous slide follows if we take ϕ to be infinity outside of $\mathcal{A}$ and zero inside. Measures of the form $e^{-\phi}dx$ with ϕ convex are called *log-concave*. Prékopa's theorem says that *marginals* of log-concave measures are log-concave. We also see that the B-M version on the previous slide holds not just for Lebesgue measure, but for any log-concave measure (like Gaussians).

Proof of Prekopa

It suffices to prove Prekopa when $n = 1$ (!) The main point in the proof we will give is the Brascamp-Lieb inequality:

Theorem

Let ψ be convex on $\mathbb{R}$ and assume

$$\int e^{-\psi} dx < \infty.$$

Let u be a function in $L^2(e^{-\psi})$, and put

$$\hat{u} = \int u e^{-\psi} / \int e^{-\psi}.$$

Then

$$\int (u - \hat{u})^2 e^{-\psi} \leq \int (u')^2 / \psi'' e^{-\psi}.$$

Remarks on Brascamp-Lieb

1. $u = \psi'$ gives equality.

2. Equivalent formulation: The minimal solution to $u' = f$ in $L^2(e^{-\psi})$ satisfies

$$\int u^2 e^{-\psi} \leq \int f^2/\psi'' e^{-\psi}.$$

3. This is similar to Hormander's L^2-estimates for the $\bar{\partial}$-equation.

Proof of Brascamp-Lieb

We assume that ψ is smooth and strictly convex. It has a minimum somewhere; say for $x = 0$. Write

$$u - u(0) = k\psi'.$$

Then $u' = k'\psi' + k\psi''$. We get

$$\int (u')^2/\psi'' e^{-\psi} = \int (k^2\psi'' + (k'\psi')^2/\psi'' + 2k'k\psi')e^{-\psi} \geq \int (k\psi')^2 e^{-\psi}$$

$$= \int (u - u(0))^2 e^{-\psi} \geq \int (u - \hat{u})^2 e^{-\psi}.$$

□

Proof of Prekopa

A direct computation, with a twist:

$$d/dt \log \int e^{-\phi} = -\frac{\int \dot{\phi} e^{-\phi}}{\int e^{-\phi}} = -\hat{\dot{\phi}}.$$

Differentiating once more we get

$$\frac{-\int \ddot{\phi} e^{-\phi} + \int (\dot{\phi})^2 e^{-\phi}}{\int e^{-\phi}} - (\hat{\dot{\phi}})^2.$$

Rewriting:

$$\frac{-\int \ddot{\phi} e^{-\phi} e^{-\phi}}{\int e^{-\phi}} + \frac{\int (\dot{\phi} - \hat{\dot{\phi}})^2 e^{-\phi}}{\int e^{-\phi}}.$$

$$\frac{-\int \ddot{\phi} e^{-\phi} e^{-\phi}}{\int e^{-\phi}} + \frac{\int (\dot{\phi} - \hat{\dot{\phi}})^2 e^{-\phi}}{\int e^{-\phi}}.$$

By Brascamp-Lieb this is smaller than

$$-\int (\ddot{\phi} - (\dot{\phi}')^2/\phi'') e^{-\phi}.$$

The integrand is the determinant of the Hessian of ϕ, divided by ϕ'', hence positive.

□

The Legendre transform

Let ϕ be any function on $\mathbb{R}^n$, taking values in $\mathbb{R} \cup \infty$. Its Legendre transform is

$$L(\phi)(y) = \hat{\phi}(y) = \sup_x y \cdot x - \phi(x).$$

Example 1: $\phi(x) = 0$. Then $\hat{\phi}(y) = \infty$ except for $y = 0$ and $\hat{\phi}(0) = 0$.

Example 2: $\phi(x) = x^2/2$. Then $\hat{\phi}(y) = y^2/2$.
These examples illustrate the idea that the Legendre transform is an analog of the Fourier-Laplace transform, if we replace integrals by suprema. If we associate to ϕ the density $e^{-\phi}$, the second example is analogous to 'the Fourier transform of a Gaussian is a Gaussian'. The first example is analogous to 'the Fourier transform of 1 is a Dirac measure'.

Let ϕ° be the supremum of all affine functions smaller than ϕ.

Theorem

$$L^2(\phi) = \phi^\circ.$$

By the hyperplane separation theorem, $\phi^\circ = \phi$ if and only if ϕ is convex and lower semicontinuous.

Corollary

$$L^2(\phi) = \phi$$

if and only if ϕ is convex and lower semicontinuous.

Proof

$$\phi^\circ(x) = sup_{y,c} y \cdot x - c.$$

The sup is taken over (y, c) such that

$$y \cdot z - c \leq \phi(z) \quad \text{for all } z$$

i. e. $\hat{\phi}(y) \leq c$. Hence

$$\phi^\circ(x) = \sup_{y,c} y \cdot x - c = \sup_{y} y \cdot x - \hat{\phi}(y) = L^2(\phi)(x).$$

□

A special case

We look at functions ϕ of class C^2, strictly convex in all of $\mathbb{R}^n$. Assume also that ϕ grows faster than linearly at infinity.

Theorem

$\hat{\phi}$ is also of class C^2, strictly convex in all of $\mathbb{R}^n$. The map

$$x \to \partial\phi(x)$$

is a diffeomorphism of $\mathbb{R}^n$ with inverse $y \to \partial\hat{\phi}$. The Hessian of $\hat{\phi}$ is the inverse of the Hessian of ϕ at corresponding points.

Remark: That the two gradient maps are inverses of each other gives an alternative definition of $\hat{\phi}$; (probably) the original definition of Legendre.

Proof

The supremum in

$$\hat{\phi}(y) = \sup_x x \cdot y - \phi(x)$$

is attained in the unique point x_y where $y = \partial\phi(x)$. Hence $\hat{\phi}(y) = x_y \cdot y - \phi(x_y)$, so $\hat{\phi}$ is at least one time continuously differentiable. Expressed slightly differently

$$x \cdot y \leq \phi(x) + \hat{\phi}(y)$$

with equality exactly when $y = \partial\phi(x)$. Since $L^2(\phi) = \phi$, equality also holds exactly when $x = \partial\hat{\phi}(y)$. Therefore $\partial\phi$ and $\partial\hat{\phi}$ are inverse maps, so in fact $\hat{\phi}$ is of class C^2. This implies also the last claim.

The differential of the Legendre transform

Theorem

The map $\phi \to L(\phi]$ is (Frechet) differentiable (on our class of functions) with derivative

$$dL_\phi.u(y) = -u \circ \partial\hat{\phi}(y)$$

if u has compact support.

In other words

$$(d/dt|_0)L(\phi + tu)(y) = -u(\partial\hat{\phi}(y)).$$

Equivalently:

$$dL_\phi.u(\partial\phi(x)) = -u(x).$$

Proof

The gradient map of $L(\phi + tu)$ is the inverse of the gradient map of $\phi + tu$. Hence it is a C^1-function of t. Therefore $L(\phi + tu)$ is also differentiable in t.
Recall that

$$\hat{\phi}(\partial\phi(x)) = x \cdot \partial\phi(x) - \phi(x).$$

Hence

$$L(\phi + tu)(\phi(x) + tu(x)) = x \cdot \partial\phi(x) + tx \cdot \partial u(x) - \phi(x) - tu(x).$$

The theorem follows by identifying terms of order 1 in t.

The space of convex functions

Let

$$CVX = \{\phi : \mathbb{R}^n \to \mathbb{R}; (\phi_{jk}) > 0\}.$$

Let also

$$T(CVX) = C_c^2(\mathbb{R}^n).$$

We introduce two Riemannian metrics on the tangent space at a point ϕ in CVX.

$$|u|_0^2 := \int_{R^n} |u|^2 dx$$

and

$$|u|_1^2 := \int_{R^n} |u|^2 MA(\phi),$$

where $MA(\phi) = \det(\phi_{jk})dx$.

We have seen that the Legendre transform maps CVX to itself.

Let $t \to \phi_t$ be a curve in CVX, and $\psi_t = L(\phi_t)$.

$$|\dot{\phi}_t|_0^2 = \int |\dot{\phi}_t|^2(x)dx = [x = \partial\psi_t(y)] = \int |\dot{\psi}_t|^2 MA(\psi_t) = |\dot{\psi}_t|_1^2.$$

Hence the Legendre transform is an isometry between the two metrics.

Connections on a Riemannian manifold

If X and Y are vector fields on a Riemannian manifold, a conection is a way to differentiate X along Y; $D_Y X$. It must satisfy the product rule

$$D_Y(fX) = fD_Y X + Y(f)X,$$

if f is a function. D is compatible with the metric if

$$Y|X|^2 = 2\langle D_Y X, X\rangle.$$

D is symmetric if $D_Y X = D_X Y$ when X and Y Lie commute. There is a unique symmetric connection, compatible with the metric on a *finite dimensional* Riemannian manifold.

Geodesics

A curve is a geodesic if its geodesic curvature is zero, i. e.

$$\text{``}((d/dt)\dot{x}_t))'' = D_{\dot{x}_t}\dot{x}_t = 0.$$

Let M be $\mathbb{R}^n$ with the trivial metric. The Riemannian connection is ($X = (X_1, ...X_n)$)

$$D_Y(X) = (Y(X_1), ...Y(X_n))$$

x_t is a geodesic if and only if

$$(d/dt)\dot{x}_t = 0 \quad x_t = x_0 + t\dot{x}_0.$$

Let ϕ_t be a curve in CVX. Then

$$\dot{\phi}_t|\dot{\phi}_t|_0^2 = (d/dt)|\dot{\phi}_t|_0^2 = 2\int \ddot{\phi}_t\dot{\phi}_t dx.$$

This suggests that the connection for our first metric should be such that

$$D_{\dot{\phi}_t}\dot{\phi}_t = \ddot{\phi}_t.$$

Geodesics are then given by $\phi_t = \phi_0 + t\dot{\phi}_t$.

Notice that between any two functions, ϕ_0 and ϕ_1 there is always a geodesic, $t\phi_1 + (1-t)\phi_0$.

Moreover, given a function ϕ and a direction in the tangents space, u, there is a short geodesic segment starting in that direction, $\phi + tu$.

What about the second metric?

A computation that we postpone gives that

$$(d/dt)\int |\dot\phi_t|^2 MA(\phi_t) = 2\int c(\phi)\dot\phi_t MA(\phi_t),$$

where

$$c(\phi_t) = \ddot\phi_t - |d\dot\phi_t|^2_{(\phi_t^{jk})}.$$

We put

$$D_{\dot\phi_t}\dot\phi_t := c(\phi_t).$$

A linear algebra exercise gives that

$$c(\phi_t) = MA(\phi(t,x))/MA(\phi_t).$$

(This is easy to see when $n = 1$.)

Hence geodesics for the second metric are given by solutions to the homogenous Monge-Ampere equation

$$MA(\phi(t,x)) = 0.$$

These are mapped to linear curves

$$\psi_t = \psi_0 + t\dot{\psi}_0$$

under the Legendre transform.

Consequences

Since the two metrics are isometric (under the Legendre transform), we still have:

1.*Between any two points, ϕ_0 and ϕ_1, there is a geodesic (for the second metric!) joining them.*
(This means we can solve the homogeneous Monge-Ampere equation with given boundary values.)

2. *Given one point ϕ and a direction in the tangent space u, there is a geodesic segment starting at ϕ in that direction.*
(Solvability of the initial value problem for the homogeneous Monge-Ampere equation.)

Hessians

Connections also act on differential forms by the product rule

$$Y(\alpha.X) = D_Y\alpha.X + \alpha.D_Y X.$$

If F is a function on M, its Hessian is the quadratic form

$$H(F)(X,X) := D_X dF.X.$$

Then

$$(d/dt)^2 F(x_t) = (d/dt)dF.\dot{x}_t = dF.D_{\dot{x}_t}\dot{x}_t + H(F)(\dot{x}_t, \dot{x}_t).$$

This gives another way to define the Hessian of F.

Let $M = CVX$ and take

$$P(\phi) = -\log \int e^{-\phi},$$

the Prekopa function on CVX.

$$(d/dt)^2 P(\phi_t) = \frac{\int (\ddot{\phi}_t - (\dot{\phi}_t - \widehat{\dot{\phi}_t})^2) e^{-\phi_t}}{\int e^{-\phi_t}} =$$

$$= dP.c(\phi_t) + \frac{\int |d\dot{\phi}_t|^2 e^{-\phi_t} - \int (\dot{\phi}_t - \widehat{\dot{\phi}_t})^2 e^{-\phi_t}}{\int e^{-\phi_t}}.$$

Conclusion

Hence the Hessian of the Prekopa function is

$$H(P) = \frac{\int |d\dot{\phi}_t|^2 e^{-\phi_t} - \int (\dot{\phi}_t - \widehat{\dot{\phi}_t})^2 e^{-\phi_t}}{\int e^{-\phi_t}},$$

the Brascamp-Lieb quadratic form. Every geodesic (for the second metric!) is convex in (t, x). (This is not true for the first metric.) Therefore, Prekopa's theorem implies that P is convex along geodesics, which in turn implies that the Hessian is positive.

This is the Brascamp-Lieb inequality in any dimension. Hence B-L is equivalent to Prekopa; they both imply each other.

The minimum principle

Proposition

Let $\phi(t,x)$ be convex in (t,x). Then

$$\inf_x \phi(t,x)$$

is a convex function of t

First proof: For any $p > 0$

$$-(1/p)\log \int_X e^{-p\phi(t,x)}dx$$

is convex in t by Prékopa.Take limit as $p \to \infty$. □

Second proof:

Let

$$E_\phi := \{(s, t, x); s > \phi(t, x)\}$$

be the epigraph of ϕ. A function is convex if and only if its epigraph is a convex set. Use that the projection of a convex set is convex.

□

Complex version ?

Let $\phi(\tau, z)$ be psh in $\mathbb{C}^{n+1}$. Put

$$\tilde{\phi}(\tau) := -\log \int e^{-\phi(\tau,z)} d\lambda(z).$$

Is $\tilde{\phi}$ psh?

No!

Kiselman's example

Take $n = 1$. Let

$$\phi(\tau, z) = |z - \bar{\tau}|^2 - |\tau|^2 = |z|^2 - 2\mathrm{Re}\, z\tau.$$

Then

$$\int e^{-\phi(\tau,z)} = ce^{|\tau|^2}.$$

Hence $\tilde{\phi}(\tau)$ is not psh.

Nevertheless, $\tilde{\phi}$ is psh under some conditions:
1. If $\phi(\tau, z) \leq C(\tau) + (n+1)\log(1 + |z|^2)$.
2. If ϕ is S^1-invariant in z; $\phi(\tau, e^{i\theta} z) = \phi(\tau, z)$.

Why?

Example 1

Theorem

Assume that $U \subset \mathbb{C}^n$ is pseudoconvex and balanced in the sense that $z \in U$ and $|\lambda| \leq 1$ implies that $\lambda z \in U$. Let $\psi(\tau, z)$ be S^1-invariant in z and psh in $\Delta \times U$, and put

$$\tilde{\psi}(\tau) = -\log \int_U e^{-\psi(\tau,z)} d\lambda(z).$$

Then $\tilde{\psi}$ is subharmonic.

Example 2

Theorem

Let ψ be psh in $\Delta \times (\mathbb{C}^*)^n$ and toric invariant in z in the sense that

$$\psi(\tau, e^{i\theta_1} z_1, ... e^{i\theta_n} z_n) = \psi(\tau, z).$$

Let as before

$$\tilde{\psi}(\tau) = -\log \int e^{-\psi(\tau, z)} d\lambda(z).$$

Then $\tilde{\psi}$ is subharmonic.

Explanation

Change variables by $z_j = e^{\zeta_j}$, $\zeta = \xi + i\eta$. Then

$$\psi(\tau, z_1, ...z_n) = \psi(\tau, e^{\xi_1}, ...e^{\xi_n}) =: \phi(\tau, \xi_1, ...\xi_n).$$

We have

$$\int e^{-\psi(\tau,z)} d\lambda(z) = \int_{\mathbb{R}^n} e^{-\phi(\tau,\xi)+\xi_1...\xi_n} d\xi.$$

If ψ is S^1-invariant in τ too, we get back Prekopa.We also get that

$$\inf_{\xi} \phi(\tau, \xi)$$

is subharmonic; Kiselman's minimum principle.

Interlude: The Bergman kernel

Let (X, μ) be a measure space; $\mu \geq 0$. Let V be a closed subspace of $L^2(X, \mu)$.

Assume that for all x in X, the evaluation map

$$ev_x(f) = f(x)$$

is bounded on V. Then there is, for all x, and f in V, an element k_x in V such that

$$f(x) = \int f(y)\overline{k_x(y)}d\mu(y).$$

By definition

$$k(y, x) = k_x(y)$$

is the *Bergman kernel* for V.

Basic properties

Let $e_1, e_2, ...$ be an orthonormal basis for V.

Proposition

$$\sum |e_j(x)|^2 = \|ev_x\|^2 < \infty.$$

$$\sum e_j(y)\overline{e_j(x)} = k(y,x), \quad k(x,y) = \overline{k(y,x)}$$

$$\int k(x,x)d\mu(x) = \dim V.$$

All of this follows from

$$k_x(y) = \sum c_j e_j(y), \quad e_k(x) = \langle e_k, k_x \rangle = \bar{c}_k.$$

Note also that

$$\int k(y,x)\overline{k(y,x)}d\mu(y) = \int |k(y,x)|^2 d\mu(y) = \|ev_x\|^2 = k(x,x).$$

Examples

Example 1: D is a domain in $\mathbb{R}^n$; V is the space of constant functions. Its dimension is 1 and $e_1 = 1/|D|^{1/2}$. Hence

$$k_x(y) = \frac{1}{|D|}.$$

Example 2: Δ is the unit ball in $\mathbb{C}^n$; μ is Lebesgue measure. V is the space of holomorphic functions.

$$k_z(\zeta) = \frac{n!}{\pi^n} \frac{1}{(1 - \zeta \cdot z)^n}.$$

Example 3: $X = \mathbb{C}^n$, $d\mu = e^{-|z|^2} d\lambda(z)$. V is the space of holomorphic functions.

$$K_z(\zeta) = \frac{e^{\zeta \cdot \bar{z}}}{\pi^n}.$$

Complex Prekopa/Brunn-Minkowski I

Theorem

Let $\mathcal{D}$ be a pseudoconvex domain in $\mathbb{C}^{n+1}$ and $\phi(\tau, z)$ a psh function in $\mathcal{D}$. Let

$$D_\tau = \{z \in \mathbb{C}^n; (\tau, z) \in \mathcal{D}\}.$$

Let for each τ, $B_\tau(z)$ be the diagonal Bergman kernel for $A^2(D_\tau, e^{-\phi(\tau,z))}$. Then

$$\log B_\tau(z)$$

is psh in $\mathcal{D}$.

(The diagonal Bergman kernel is $k(x) = k(x, x) = \|ev_x\|^2$.)

Example 1

Let $\mathcal{D}$ be $\mathbb{C}^{n+1}$ and assume ϕ satisfies

$$\phi(\tau, z) \leq C(\tau) + (n+1)\log(1 + |z|^2).$$

Then A^2_τ consists only of constants. Hence

$$B_\tau(z) = \left(\int_{\mathbb{C}^n} e^{-\phi_\tau} d\lambda\right)^{-1/2}.$$

Hence $\tilde{\phi}(\tau)$ is subharmonic in this case.

Example 2

Assume D_τ is balanced and that ϕ_τ is S^1-invariant for all τ. Then

$$B_\tau(0) = \left(\int_{D_\tau} e^{-\phi_\tau} d\lambda\right)^{-1/2}.$$

Hence $\tilde{\phi}$ is subharmonic in this case.

Hormander's theorem

Hormander's theorem is the main ingredient in the proof of the complex Prekopa theorem. It is a complex analog of the Brascamp-Lieb inequality (actually proved earlier).

Theorem

Let D be a pseudoconvex domain in $\mathbb{C}^n$ and let ϕ be smooth and strictly psh in D. Let f be a $\bar\partial$-closed $(0,1)$-form in D. Then the L^2-minimal solution to $\bar\partial u = f$ satisfies the estimate

$$\int_D |u|^2 e^{-\phi} \leq \int_D |f|^2_{\partial\bar\partial\phi} e^{-\phi},$$

where

$$|f|^2_{\partial\bar\partial\phi} = \sum \phi^{j\bar{k}} f_j \bar{f}_k.$$

Geometric form of Hormander's theorem

Let X be a compact complex manifold and L a holomorphic line bundle over X equipped with a strictly positively curved metric $h = e^{-\phi}$. Let f be a $\bar\partial$-closed $(n,1)$-form on X with values in L.

Theorem

Let u be the L^2-minimal solution to $\bar\partial u = f$. Then

$$\int_X |u|^2 e^{-\phi} \leq \int_X |f|^2_{\partial\bar\partial\phi} e^{-\phi}$$

1. u is an $(n,0)$-form so its L^2-norm $\int c_n u \wedge \bar u$ is well defined *without choosing any measure on* X.

2. f can be written locally $f = f_0 \wedge v$ where v is $(n,0)$ and f_0 is $(0,1)$.

$$|f|^2_{\partial\bar\partial\phi} =: c_n v \wedge \bar v |f_0|^2_{\partial\bar\partial\phi},$$

where the last factor is the pointwise norm of f_0 w r t the Kahler metric $i\partial\bar\partial\phi$.

Sketch of proof of complex Prekopa

Assume first that $\mathcal{D} = \Delta \times D$ is a cylinder; $D\tau = D \subset \mathbb{C}^n$. Let

$$\partial_\tau^\phi = e^\phi \partial/\partial\tau e^{-\phi} = \partial_\tau - \partial_\tau \phi.$$

Since, if h is holomorphic

$$h(z) = \int h(\zeta)\overline{K_\tau(\zeta, z)} e^{-\phi(\tau,\zeta)},$$

we get

$$\partial_\tau^\phi K_\tau(\cdot, z) \perp h$$

for any holomorphic h.

We have

$$K_\tau(z,z) = \int K_\tau(\zeta,z)\overline{K_\tau(\zeta,z)}e^{-\phi}.$$

It follows (using the orthogonality condition)

$$\partial_{\bar\tau} K_\tau = \int \partial_{\bar\tau} K_\tau(\zeta,z)\overline{K_\tau(\zeta,z)}e^{-\phi}$$

and

$$\partial_\tau \partial_{\bar\tau} K_\tau = \int \partial_\tau^\phi \partial_{\bar\tau} K_\tau(\zeta,z)\overline{K_\tau(\zeta,z)}e^{-\phi} + pos.$$

Use the commutator relation

$$\partial_\tau^\phi \partial_{\bar\tau} = \partial_{\bar\tau}\partial_\tau^\phi + \phi_{\tau\bar\tau}.$$

The result is

$$\partial_\tau \partial_{\bar\tau} K_\tau \geq \int \partial_{\bar\tau} \partial_\tau^\phi K\tau \overline{K_\tau} e^{-\phi} + \int \phi_{\tau\bar\tau} |K_\tau|^2 e^{-\phi} =: I + II$$

But, using again the orthogonality

$$I = -\int |\partial_\tau^\phi K_\tau|^2 e^{-\phi} =: -\int |u|^2 e^{-\phi}.$$

This term has a bad sign, but we know that u is orthogonal to all holomorphic functions. So, we can use Hormander's inequality

$$\int |u|^2 e^{-\phi} \leq \int_D |\bar\partial_\zeta u|^2_{\partial\bar\partial\phi} e^{-\phi}.$$

And

$$\bar\partial_\zeta u = -\bar\partial_\zeta \partial_\tau \phi(\tau, \zeta) K_\tau.$$

Putting things together

we get that

$$\partial_\tau \partial_{\bar\tau} K_\tau \geq \int [\phi_{\tau\bar\tau} - |\bar\partial_\zeta \partial_\tau \phi|^2_{\partial\bar\partial\phi}] |K_\tau|^2 e^{-\phi} \geq 0.$$

The last inequality follows since

$$c(\phi) := \phi_{\tau\bar\tau} - |\bar\partial_\zeta \partial_\tau \phi|^2_{\partial\bar\partial\phi} = \frac{MA_{\tau,\zeta}(\phi)}{MA_\zeta(\phi)}.$$

This shows that $K_\tau(z,z)$ is subharmonic in τ for z fixed. How do we see that $\log K_\tau$ is psh?

Replace ϕ by $\phi + \psi(\tau)$, with ψ subharmonic. It follows that $e^\psi(\tau)K_\tau$ is subharmonic for any such ψ. This implies that $\log K_\tau$ is subharmonic.

To see that $\log K_\tau(z,z)$ is psh in (τ, z) we give an extension of the theorem.

More general version

Let $\mathcal{D}$ be as before and consider for each fiber D_τ a compactly supported complex measure μ_τ in D_τ.

We say that μ_τ is holomorphic in τ if

$$\tau \to \int_{D_\tau} h(\tau, z) d\mu_\tau(z)$$

is holomorphic for each h holomorphic in $\mathcal{D}$ near D_τ.

Example: $\mu_\tau = \delta_z$, a Dirac mass at a fixed point z. Or, $\mu_\tau = \delta_{f(\tau)}$ where f is holomorphic.
Let

$$\|\mu_\tau\|_\tau := \sup_{|h|_\tau \leq 1} |\int_{D_\tau} h(\tau, z) d\mu_\tau(z)|$$

In the first example

$$\|\mu_\tau\|_\tau^2 = K_\tau(z, z).$$

In the second example

$$\|\mu_\tau\|_\tau^2 = K_\tau(f(\tau), f(\tau)).$$

Theorem

Under the same assumptions as before, if μ_τ is holomorphic,

$$\tau \to \log \|\mu_\tau\|$$

is subharmonic.

This implies that $\log K_\tau(z, z)$ is psh in $\mathcal{D}$ – and many other things.
The proof is basically the same.

Domains that are not cylinders

We may assume that the domain $\mathcal{D}$ is strictly pseudoconvex and smoothly bounded, since any pseudoconvex set can be exhausted by such domains. (If $\Omega = \cup \Omega_j$ where Ω_j is an increasing family of relatively compact subdomains; K_Ω is a decreasing limit of K_{Ω_j}. Then $\mathcal{D} = \{(\tau, z); \rho(\tau, z) < 0\}$ where ρ is smooth, psh and extends a bit across the boundary. Localizing around a fiber D_0, we can find a cylinder $\mathcal{D}' := \Delta \times U$ which contains $\mathcal{D} \cap (\Delta \times \mathbb{C}^n)$ where ρ is defined and psh.

Let $k_j(s)$ be an increasing family of convex functions on $\mathbb{R}$, all equal to zero on the negative half-axis and tending to infinity for $s > 0$. Let

$$\phi_j = \phi + k_j \circ \rho.$$

The crux of the matter is to prove that $K_{U,\phi_j(\tau,\cdot)}$ increases to $K_{D_\tau,\phi}$. The crucial step is an approximation result:

Theorem

Let U be pseudoconvex and ρ smooth, exhaustive and psh in U. Let

$$V = \{\rho < 0\}.$$

Then any holomorphic function in V in L^2 can be approximated by holomorphic functions in U in the L^2-sense.

The difficulty is that V may not be smoothly bounded.

Interpretation of the 'more general version'

Theorem

Under the same assumptions as before

$$\tau \to \log \|\mu_\tau\|$$

is subharmonic.

Given $\mathcal{D}$ we can think of the family of Hilbert spaces

$$A^2_\tau = A^2(D_\tau, \phi) = \{h \in H(D_\tau); \int_{D_\tau} |h|^2 e^{-\phi(\tau,\cdot)} < \infty\}$$

as a bundle of Hilbert spaces, over the base – a subset of the τ-axis. The measures μ_τ define a holomorphic section of the dual bundle.

If the norms of any holomorphic section of a vector bundle are log-subharmonic; the bundle is *negatively curved*. Hence we have – roughly that the dual of the bundle A^2_τ is negatively curved. This means that the bundle itself is *positively curved*.

Review of holomorphic vector bundles and their curvature

Locally a holomorphic vector bundle of rank r is $E = U \times \mathbb{C}^r$, U domain in $\mathbb{C}^n$. A holomorphic section is

$$s = (s_1(z), ...s_r(z)) = \sum s_j e_j.$$

A hermitian metric is given by a matrix-valued function $A = (a_{j\bar{k}})$; hermitian and positive definite. A connection on E maps sections to E-valued 1-forms, ($s \to Ds$) satisfying

$$Dfs = df \otimes s + fDs$$

if f is a function.

This means that

$$Ds = D\sum s_j e_j = \sum ds_j \otimes e_j + \sum s_j De_j = \sum ds_j \otimes e_j + \sum s_j \omega_{jk} e_k,$$

$$D = d + \omega.$$

(ω is a matrix of 1-forms). The curvature of the connection is an endomorphism-valued 2-form

$$\Theta s = D^2 s.$$

One important fact is that, $\Theta = D^2$ is a differential operator of order 0:

$$\Theta fs = D(df \otimes s + fDs) = df \wedge Ds + fD^2 s - df \wedge Ds = f\Theta s.$$

So, $D = d + \omega$ and

$$\Theta s = \sum \Theta_{jk} s_k e_j.$$

One verifies that

$$\Theta = d\omega + \omega \wedge \omega.$$

By definition, D is compatible with the complex structure if ω is of bidegree $(1, 0)$. Moreover, D is said to be compatible with the metric if

$$d\langle s, s' \rangle = \langle Ds, s' \rangle + \langle s, Ds' \rangle.$$

Theorem

There is exactly one connection which is compatible with both the metric and the complex structure. (It is called the 'Chern connection'.)

Indeed, D must satisfy

$$\partial(sAs^\dagger) = \partial(s)As^\dagger + s\partial(A)s^\dagger,$$

if s is holomorphic. Hence

$$\omega = A^{-1}\partial A$$

and

$$\Theta = \bar{\partial}\omega + \partial\omega + \omega \wedge \omega = \bar{\partial}\omega,$$

a $(1,1)$-form. (Use that $A\omega = \partial A$, so

$$\partial A \wedge \omega + A\partial\omega = 0,$$

whence

$$\omega \wedge \omega + \partial\omega = 0$$

Example: A line bundle ($r = 1$): $A = e^{-\phi}$,

$$\omega = A^{-1}\partial A = -\partial\phi, \quad \Theta = -\bar{\partial}\partial\phi = \partial\bar{\partial}\phi.$$

□

We next look at a consequence of metric compatibility. Let s be a holomorphic section. Then

$$\partial\langle s, s\rangle = \langle D^{1,0}s, s\rangle$$

and

$$\bar{\partial}\partial\langle s, s\rangle = \langle\bar{\partial}D^{1,0}s, s\rangle - \langle D^{1,0}s, D^{1,0}s\rangle.$$

Hence

$$\partial\bar{\partial}\langle s, s\rangle = -\langle\Theta s, s\rangle + \langle D^{1,0}s, D^{1,0}s\rangle.$$

$$\partial\bar{\partial}\langle s, s\rangle = -\langle\Theta s, s\rangle + \langle D^{1,0}s, D^{1,0}s\rangle.$$

Definition: The Chern curvature is positive in the sense of Griffiths if

$$i\langle\Theta s, s\rangle$$

is a positive $(1, 1)$-form for any s. Negativity is defined in an analogous way.

$\square$

Proposition

The Chern curvature is negative if and only if the function

$$\langle s, s\rangle$$

is psh in U for any holomorphic s. This is also equivalent to

$$\log\langle s, s\rangle$$

psh.

proof

It is clear from the formula on top of the previous slide that negativity implies that $\langle s, s \rangle$ is psh. Conversely, this implies negativity since we may choose s so that $D^{1,0}s = 0$ at any given point.
Since we may multiply s by any holomorphic function, this implies that even its log is psh.

Dual bundles

Given a vector bundle E, its dual bundle E^* is again a vector budle whose fibers are the duals of the fibers of E. In our simplified local picture, E^* is again $U \times \mathbb{C}^r$, but with a different norm:

$$\|t\|_z = \sup_{\|s(z)\|=1} |s \cdot t(z)|.$$

One verifies that the curvature of the dual is negative if and only if the curvature of the bundle is positive (check the case $r = 1$!).

Complex Prekopa III; ‘positivity of direct images’

Let $\mathcal{X}$ be a complex Kahler manifold and

$$p : \mathcal{X} \to Y$$

be a (holomorphic) smooth proper fibration of relative dimension n. This means that p is surjective, has surjective differential and all fibers $X_Y := p^{-1}$ are smooth manifolds. Let $(L, e^{-\phi})$ be a holomorphic hermitian line bundle over $\mathcal{X}$ with semipositive curvature $i\partial\bar{\partial}\phi \geq 0$. For each y in the base, let

$$E_y = \{u \in H^{n,0}(X_y, L)\}$$

equipped with the metric

$$\|u\|^2 := c_n \int_{X_y} u \wedge \bar{u} e^{-\phi}.$$

Then...

Theorem

E_y are the fibers of a holomorphic vector bundle over Y, E. This vector bundle, with its metric, has nonnegative curvature (in the sense of Griffiths and even in the stronger sense of Nakano). Along any complex one-dimensional curve in the base

$$i\langle \Theta u, u\rangle_y \geq c_n \int_{X_y} c(\phi) u \wedge \bar{u} e^{-\phi},$$

where, as before,

$$\text{``}c(\phi) := \frac{MA_{\tau,\zeta}(\phi)}{MA_{\zeta}(\phi)}\text{''}.$$

Terminology

On $\mathcal{X}$ we have the sheaf, $\mathcal{F}$, of fiberwise holomorphic $(n, 0)$-forms, or equivalently, the sheaf of sections of $K_{\mathcal{X}/Y} + L$, where $K_{\mathcal{X}/Y} = K_{\mathcal{X}} - p^* K_Y$ is the *relative canonical bundle*. Given a sheaf $\mathcal{F}$ over $\mathcal{X}$, we get a direct image sheaf

$$p_*(\mathcal{F})$$

on the base Y. The sections of the direct image over an open set U in the base are by definition the sections of $\mathcal{F}$ over $p^{-1}(U)$. In this case, the direct image sheaf is 'locally free', meaning that it is the sheaf of sections of a vector bundle. This vector bundle is E. Abusing language

$$E = p_*(K_{\mathcal{X}/Y} + L).$$

Griffiths theorem

Griffiths proved the positivity of direct images in the case when L is trivial, with a different method. His motivation was also different, coming from generalization of period maps on Riemann surfaces.

Note first that all fibers X_y are diffeomorphic (which is not the case in the non-proper setting!), but not biholomorphic. One is studying the variations of complex structure on a fixed smooth manifold.

Cylinder case

We assume $\mathcal{X} = X \times \Delta$ and that L is the pullback of a bundle over X; so the same on all fibers. We can think of the metric on L over $\mathcal{X}$ as a (complex) curve of metrics

$$\tau \to \phi_\tau$$

with $\tau \in \Delta$. If $\tau \to u_\tau$ is a holomorphic section of E

$$\partial_\tau \langle u_\tau, u_\tau \rangle = \int_X \partial_\tau^\phi u_\tau \wedge \bar{u}_\tau e^{-\phi_\tau}.$$

Hence, the connection on E should be $D^{1,0} u_\tau = \partial_\tau^\phi u_\tau$, and

$$\omega u_\tau = -(\partial_\tau \phi_\tau) u_\tau,$$

or *rather its projection on holomorphic forms*; a Toeplitz operator.

$$(\partial^2/\partial\tau\partial\bar{\tau})\langle u_\tau, u_\tau\rangle = -c_n\int_X \phi_{\tau\bar{\tau}} u\wedge\bar{u}e^{-\phi} + \|\dot{\phi}_\tau u\|^2.$$

Decompose the last term in two orthogonal parts:

$$\|\dot{\phi}_\tau u\|^2 = \|D^{1,0}u\|^2 + \|(\dot{\phi}_\tau)^\perp\|^2.$$

If $D^{1,0}u = 0$ at one given τ

$$(\partial^2/\partial\tau\partial\bar{\tau})\langle u_\tau, u_\tau\rangle = -c_n\int_X \phi_{\tau\bar{\tau}} u\wedge\bar{u}e^{-\phi} + \|(\dot{\phi}_\tau u)^\perp\|^2 = -\langle\Theta u, u\rangle.$$

Using Hormander's theorem to estimate the orthogonal part, we get the theorem.

Philosophical interpretation

Consider the bundle E as a subbundle of the bundle F with fibers

$$L^2_{(n,0)}(X, e^{-\phi_\tau})$$

(non-holomorphic $(n, 0)$-forms). The first term in the curvature formula

$$c_n \int_X \phi_{\tau\bar{\tau}} u \wedge \bar{u} e^{-\phi}$$

is the curvature of F. The second

$$\|(\dot{\phi}_\tau u)^\perp\|^2$$

is the second fundamental form of the subbundle E in F.
The curvature of E is bounded from below by the Toeplitz operator with symbol $c(\phi)$.

Mabuchi space

$$\mathcal{M}_L(X) = \{\phi; e^{-\phi} \quad \text{metric on } L;\ \omega_\phi := i\partial\bar{\partial}\phi > 0\}$$

This is our analog of the space of convex functions in $\mathbb{R}^n$.

$$T_\phi(\mathcal{M}) = C^\infty(X, \mathbb{R}).$$

Norm on the tangent space:

$$\|u\|_\phi^2 := \int_X |u|^2 \omega_\phi^n / n!.$$

Let $t \to \phi_t$ be a curve in $\mathcal{M}$. To find the Riemannian connection on $\mathcal{M}$ we look at

$$2\langle D_{\dot\phi}\dot\phi, \dot\phi\rangle = (d/dt)\|\dot\phi\|^2 =$$

$$2\int_X \ddot\phi\dot\phi\omega_\phi^n/n! + \int_X |\dot\phi|^2 i\partial\bar\partial\dot\phi \wedge \omega_\phi^{n-1}/(n-1)!.$$

In the last term we can integrate by parts (ϕ is not a function, but $\dot\phi$ is a function) and get

$$-2\int_X \dot\phi[\ddot\phi - |\partial\dot\phi|^2_{\partial\bar\partial\phi}]\omega_\phi^n/n!.$$

Hence (?)

$$D_{\dot\phi}\dot\phi = c(\phi).$$

Let, for $\tau = t + is$,

$$\phi(\tau, z) = \phi_\tau(z) = \phi_t(z).$$

It is defined for τ in a strip. We see that ϕ_t is a geodesic if and only if $\phi(\tau, z)$ lies in $\mathcal{M}$ for any fixed τ, is psh, and satisfies the homogeneous complex Monge-Ampere equation.

We say that ϕ_t is a *generalized geodesic* if it is psh, locally bounded, and satisfies the HCMA. It is a *subgeodesic* is it is locally bounded and psh.

A theorem of X.X. Chen (... and Blocki, Tosatti-Weinkove) says that any two points in $\mathcal{M}$ can be connected with a generalized geodesic of class $C^{1,1}$

Let $\underline{E} = H^0(X.L)$. Let

$$\mathcal{H}$$

be the space of Hilbert norms on $\underline{E}$ (i.e. Hermitian $(n \times n)$ matrices, given a base). For any $\phi \in \mathcal{M}$ we get an element in $\mathcal{H}$,

$$\|u\|_\phi^2 := \int |u|^2 e^{-\phi} \omega_\phi^n / n!.$$

This map from $\mathcal{M}$ to $\mathcal{H}$ is called Hilb; $\phi \to Hilb(\phi)$.

We also get a map in the opposite direction, called FS (for Fubini-Study). This is essentially the Bergman kernel.

One can think of the map Hilb as a counterpart of the Legendre transform, and FS as the inverse Legendre transform.

The FS-map

Let $\| \cdot \| \in \mathcal{H}$. Then we get an element in $\mathcal{M}$; ψ, by

$$|u|^2 e^{-\psi}(x) := \frac{|u|^2(x)}{\sup_{\|u'\|^2=1} |u'|^2(x)}.$$

In other words

$$\psi(x) = \log \sup_{\|u'\|^2=1} |u'|^2(x).$$

Notice that the Bergman kernel is not a function, but a metric on L.

The idea is that (if we replace L by kL and let $k \to \infty$; $FS \circ Hilb$ should go to the identity map. This is a sort of approximation of our infinite dimensional manifold $\mathcal{M}$ by finite dimensional objects.

Bergman kernel asymptotics

...due to Bouche, Tian, Zelditch, Catlin...

Let $\phi \in \mathcal{M}$. Let

$$B_{k\phi}(x) = \sup_{\|u'\|^2_{k\phi}=1} |u'|^2(x),$$

i.e.

$$\log B_{k\phi} = FS \circ Hilb(k\phi).$$

Theorem

$$\lim_{k\to\infty} B_{k\phi} e^{-k\phi} = \pi^{-n}.$$

Bergman kernel asymptotic II

Instead of looking at quantization via the Hilbert space $H^0(X, L)$, we can look at $H^{n,0}(X, L) = H^0(X, L + K_X)$ (as before). The Bergman kernel is then a metric on $L + K_X$. Equivalently it is a metric on L, times a volume form.

Theorem

In this setting

$$\lim \frac{B_{k\phi} e^{-k\phi}}{d_k} = \frac{1}{V(L)} (\omega_\phi)^n / n!.$$

Here $d_k = \dim H^0(X, kL + K_X)$ and $V(L)$ is the volume of L, the constant making the integral of the RHS equal to one.

Tsuji's theorem

If K_X is positive we can take $L = K_X$. Starting with an arbitrary positive metric ϕ_1 on K_X, we get a Bergman kernel metric ϕ_2 on $2K_X$. Iterating we get a sequence of metrics ϕ_k on kK_X.

Theorem

$$lim_{k\to\infty}\phi_k/k = \phi_{KE},$$

the Kahler-Einstein metric on X.

Idea of proof (first Bergman kernel)

Take a point x in X. Choose local coordinates z, centered at x, and a local frame of L such that

$$\phi = |z|^2 + O(|z|^3).$$

Rescale by $z = \zeta/\sqrt{k}$. A small ball around x, $|z| < \epsilon$ corresponds to a large ball, $\zeta < \epsilon\sqrt{k}$. Moreover

$$k\phi(z) = k\phi(\zeta/\sqrt{k}) = \||\zeta|^2 + O(1/\sqrt{k}),$$

and

$$\omega_{k\phi}^n/n! \to d\lambda.$$

The theorem says that, at x, the Bergman kernel tends to the Bergman kernel of $\mathbb{C}^n$ with measure $e^{-|\zeta|^2}d\lambda$, at the origin.

One more vector bundle

A complex curve $\tau \to \phi_\tau$ gives rise to a metric on the vector bundle

$$E = \underline{E} \times U.$$

Warning:This is not the same vector bundle as before - no $(n, 0)$**-forms!** Strangely, we have:

Theorem

Let ϕ_τ be a (generalized) complex geodesic, i.e. satisfy the HCMA. Then the induced metric on E has **negative** *curvature.*

This is actually a lot simpler than the previous results.

proof

We need to prove that if u_τ is a holomorphic section, then $\tau \to \|u_\tau\|_\tau^2$ is subharmonic.

$$\|u_\tau\|_\tau^2 = p_*(|u_\tau|^2 e^{-\phi}(\partial\bar{\partial}\phi)^n/n!),$$

where we consider all objects as defined on the total space $\mathcal{X} = X \times U$ and p is the projection on U. P_* is the push-forward – it commutes with ∂ and $\bar{\partial}$.

$$\|u_\tau\|_\tau^2 = p_*(|u_\tau|^2 e^{-\phi}(\partial\bar\partial\phi)^n/n!),$$

Hence

$$\bar\partial\|u_\tau\|_\tau^2 = p_*(u_\tau\overline{\partial^\phi u_\tau} \wedge e^{-\phi}(\partial\bar\partial\phi)^n/n!),$$

and

$$\partial\bar\partial\|u_\tau\|_\tau^2 = p_*(\partial^\phi u_\tau \wedge \overline{\partial^\phi u_\tau} \wedge e^{-\phi}(\partial\bar\partial\phi)^n/n!) + p_*(u_\tau\overline{\bar\partial\partial^\phi u_\tau} \wedge e^{-\phi}(\partial\bar\partial\phi)^n/n!)$$

Since u_τ is holomorphic, $\partial^\phi\bar\partial u_\tau = 0$. Commuting the operators, the last term is

$$-p_*(u_\tau\overline{u_\tau} \wedge \partial\bar\partial\phi \wedge e^{-\phi}(\partial\bar\partial\phi)^n/n!).$$

But, this is zero, since ϕ_τ solves the HCMA.

Notice that we need a geodesic, whereas before subgeodesic was enough.

Donaldson does not (?) consider the negatively curved bundle E, but only its determinant bundle. This is a line bundle over U, hence trivial and we can think of its metric as a function

$$\psi = \log\det(\|\cdot\|^2).$$

He proves that

$$\log\det(Hilb(k\phi))$$

tends to the *Aubin-Yau energy* or *Monge-Ampere energy* of ϕ.

Aubin-Yau energy

$\mathcal{E}(\phi)$ is defined (up to a constant) by

$$(d/dt)\mathcal{E}(\phi_t) = -\int_X \dot{\phi}\omega_\phi^n/n!/V(L).$$

More explicitly, if ϕ_0 and ϕ_1 are two elements of $\mathcal{M}$

$$\mathcal{E}(\phi_1, \phi_0) = \int_X (\phi_0 - \phi_1) \sum_0^n \omega_0^k \wedge \omega_1^{n-k}.$$

The counterpart in the case of bounded domains is

$$\int_D -\phi\omega_\phi^n/n!,$$

for ϕ that vanish on the boundary.

The other quantization

We can do exactly the same thing for ‘my’ quantization, and get another function $\log\det$. It satisfies the same convergence result; it also tends to the MA-energy.

Donaldson’s result implies that $\mathcal{E}$ is convex along geodesics. My result implies that it is concave. Hence $\mathcal{E}$ is linear along geodesics (not hard to see directly).

Fano manifolds

A compact manifold X is Fano if the canonical bundle K_X is negatively curved. Then we can take (in my theorem on direct images) $L = -K_X$ and are led to consider

$$H^0(X, -K_X + K_X) = \mathbb{C}.$$

It has a basis element $u = 1$ with norm

$$\|1\|^2_\phi = \int_X e^{-\phi}.$$

Theorem

$$\log \int_X e^{-\phi}$$

is concave along subgeodesics.

The Ding functional

is defined by

$$\mathcal{D}(\phi) = -\log \int_X e^{-\phi} + \mathcal{E}(\phi).$$

Its critical points are given by

$$\frac{\int \dot{\phi} e^{-\phi}}{\int e^{-\phi}} = \frac{\int \dot{\phi} \omega_\phi^n / n!}{V},$$

for all $\dot{\phi}$. Hence

$$e^{-\phi} = C\omega_\phi^n.$$

This is the Kahler-Einstein equation. The Ding functional is convex along geodesics.

The existence problem for KE-metrics in the Fano case amounts to proving that there are critical points of $\mathcal{D}$. This means roughly that $\mathcal{D}$ tends to infinity at infinity. A theorem of Chen-Donaldson-Sun (Tian) says that this is so if and only if $X, -K_X$ is 'stable' in a certain sense.

The uniqueness for KE-metrics in the Fano case is due to Bando and Mabuchi:

Theorem

Let $\omega_{\phi_0} = \omega_0$ and $\omega_{\phi_1} = \omega_1$ be two KE-metrics on the Fano manifold X. Then there is a holomorphic vector field V with flow F_τ, such that

$$F_1^*(\omega_1) = \omega_0.$$

Thus KE-metrics are unique modulo $Aut^0(X)$, the identity component of the automorphism group.

comparison with negatively curved case

Let X have $K_X > 0$. If ϕ is a metric on K_X; e^ϕ can be interpreted as a volume form on X. Then the equation

$$\omega_\phi = -Ric(\omega_\phi) = i\partial\bar{\partial}\log(\omega_\phi^n),$$

means that

$$e^\phi = \omega_\phi^n$$

(adjusting constants). Say ϕ_0 and ϕ_1 are two solutions and look at $\phi_1 - \phi_0$; a function. It has a max at some $x \in X$, where $i\partial\bar{\partial}(\phi_1 - \phi_0) \leq 0$. Then, from the equation, $\phi_1 - \phi_0 \leq 0$ at a max, hence everywhere.

Reversing the role of ϕ_1 and ϕ_0, we see that $\phi_1 = \phi_0$. Hence we have abslolute uniqueness.

sketch of proof

Connect ϕ_1 and ϕ_0 with a geodesic. Then $\mathcal{D}(\phi_t)$ is linear along a geodesic.

Theorem

Let ϕ_t be a general (bounded) geodesic. Assume

$$\log \int_X e^{-\phi_t}$$

is linear. Then there is a holomorphic vector field V, with flow F_t such that $F_t^*(\omega_t) = \omega_0$.

Obviously this implies BM. Robert Berman gave the first proof of BM by using the thm in the smooth case. I will sketch the proof of thm in the smooth case, assuming moreover that there are no non-trivial holomorphic fields.

Auxiliary result

Theorem

Let L be a holomorphic line bundle over X with a positively curved metric ϕ. Let f be a $\bar{\partial}$-closed $(n,1)$-form with values in L.
Let u_0 be the minimal solution to $\bar{\partial}u = f$. Assume $\|u_0\| = \|f\|$. Then there is a holomorphic $(n-1,0)$-form v such that

$$f = v \wedge \omega_\phi.$$

The converse also holds.

proof

There is a $\bar\partial$-closed $(n, 1)$-form α, such that

$$u_0 = \bar\partial^* \alpha.$$

Then

$$\bar\Box \alpha = f,$$

where

$$\bar\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

Recall

$$\Box = D^{1,0}(D^{1,0})^* + (D^{1,0})^* D^{1,0}.$$

The fundamental Kodaira-Nakano identity says

$$\bar\Box = \Box + \omega \wedge \Lambda$$

(Λ is the adjoint of multiplication with ω).

This implies that all eigenvalues of the elliptic operator $\bar{\Box}$ are greater than or equal to 1. Let e_j be a basis of eigenforms $((n, 1))$ with eigenvalues λ_j.

$$\alpha = \sum \alpha_j e_j, \quad f = \sum f_j e_j.$$

Since $\bar{\Box}\alpha = f$, $f_j = \lambda_j \alpha_j$.

$$\|u_0\|^2 = \langle \bar{\Box}\alpha, \alpha \rangle = \sum \lambda_j |\alpha_j|^2.$$

$$\|f\|^2 = \sum \lambda_j^2 |\alpha_j|^2.$$

Hence, if $\|u_0\|^2 = \|f\|^2$, α is an eigenform with eigenvalue 1, and $\alpha = f$. Moreover $\Box\alpha = 0$, which gives $(D^{1,0})^*\alpha = 0$. Since $*\alpha = v, v \wedge \omega = \alpha$, this gives the theorem.

If we take $L = -K_X$ in the previuos theorem, we get a $-K_X$-valued $(n-1, 0$-form. This can be identified with a $(1,0)$ vector field. If there are no holomorphic vector fields except 0, $v = 0$. I. e., equality never holds in Hörmander's theorem.

Now recall the proof of convexity of

$$t \to -\log \int_X e^{-\phi_t}.$$

It involved an applications of Hörmander's estimate. If equality never holds, we get strict convexity.

□

Additional wonkish remark

Look at a general positive line bundle over X, and the positivity of direct images in that case. We applied Hormander's theorem to the equation

$$\bar{\partial} u = \bar{\partial}\dot{\phi} \wedge s =: f$$

where $s \in H^0(X, L + K_X)$. Define

$$\mathcal{E}(\dot{\phi}, s) := \|f\|^2 - \|u_0\|^2 \geq 0.$$

($\mathcal{E}$ is the 'error' in Hormander's estimate).

For fixed s this is a quadratic form in $\dot{\phi} \in T_\phi(\mathcal{M})$. For fixed $\dot{\phi}$ it is a quadratic form in s.

A curvature tensor?

Geodesics in Mabuchi space.

Let $\phi \in \mathcal{M}(X; L)$. Recall that

$$FS \circ Hilb(k\phi)/k \to \phi,$$

by Bergman kernel asymptotics.

$\mathcal{H}$ is a symmetric space and as such has a natural metric. Fixing a basis in $H^0(X; L)$ we can identify elements in $\mathcal{H}$ with hermitian matrices. A curve A_t in $\mathcal{H}$ is then a geodesic if

$$\frac{d}{dt} A^{-1} \dot{A} = 0$$

As before we think of $A_t = A_\tau$, $\tau = t + is$, a complex curve independent of s.

Then a curve corresponds to a vector bundle metric on $E = H^0(X, L) \times U$, U a strip in the complex plane. The geodesic equation says that this metric has zero curvature.

Convergence of geodesics

Let ϕ_0, ϕ_1 be elements in $\mathcal{M}$, and ϕ_t a geodesic connecting them. Let

$$Hilb(e^{-k\phi_0}) = A_0^k, \quad Hilb(e^{-k\phi_1}) = A_1^k.$$

Connect them with a geodesic A_t^k in $\mathcal{H}_k$. Then we have (Phong-Sturm, B)

Theorem

$$\lim FS(A_t^k)/k = \phi_t,$$

uniformly at the rate $\log(k)/k$.

sketch of proof

We replace $H^0(X, kL)$ by $H^0(X, kL + K_X)$. This simplifies and implies the original version. Put

$$\psi_{t,k} := FS(A_t^k)/k.$$

Then $\psi_{t,k}$ is close to ϕ_t for $t = 0, 1$, by Bergman kernel asymptotics.

Moreover, $\psi_{t,k}$ is a subgeodesic. Since ϕ_t is the max of all subsolutions, we get roughly

$$\psi_{t,k} \leq \phi_t.$$

For the opposite direction we use an auxiliary result.

A maximum principle

Theorem

Let A_τ and B_τ be two metrics on a vector bundle over U; a domain in $\mathbb{C}$. Assume A_τ has zero curvature and B_τ has positive curvature, and that

$$A_\tau \leq B_\tau$$

on the boundary of U. Then

$$A_\tau \leq B_\tau$$

in U.

(Check in the line bundle case!)

end of proof

By positivity of direct images and the theorem on the previous slide

$$A_t \leq Hilb(k\phi_t).$$

Hence

$$FS(A_t) \geq FS \circ Hilb(k\phi_t).$$

On the other hand, we have, essentially by Bergman kernel asymptotics, that

$$FS \circ Hilb(k\phi_t)/k \geq \phi$$

modulo a small error. This gives the opposite direction.

Schwarz symmetrization

Let D be a domain in $\mathbb{R}^N$; $f : D \to \mathbb{R}$. Its Schwarz symmetrization is a radial function

$$\hat{f}(x) = \phi(|x|)$$

(with ϕ increasing) that is *equidistributed* with f.

This means that

$$\sigma_f(r) := |\{f < t\}| = \sigma_{\hat{f}}(t)$$

for all t.

Equivalently, for any measurable F

$$\int_D F(f)dx = \int_B F(\hat{f})dx.$$

The Polya-Szego theorem

Note that $\hat{f}$ is defined in a ball of the same volume as D.

Theorem

$$\int_D |\nabla \hat{f}|^p \leq \int_B |\nabla f|^p$$

for $p \geq 1$.

This is related to the isoperimetric inequality. The isoperimetric inequality is used in the proof.

Corollary

If for any radial function vanishing on the boundary or having mean zero,

$$(\int_B |f|^q)^{1/q} \leq C(\int_B |\nabla f|^p)^{1/p},$$

then the same thing holds for any function in D.

We can also look at

$$\int_D e^f$$

instead of L^q-norms. This leads to Moser-Trudinger inequalities.

Complex version?

We consider a domain in $\mathbb{C}^n$ and the Monge-Ampere energy

$$\mathcal{E}(f) = \int_D -f(i\partial\bar{\partial}f)^n/n!$$

where f vanishes on the boundary and is psh. Is there a Polya-Szego theorem in this setting?

Q1: Is $\hat{f}$ psh if f is? No!

Things work better when D is balanced and f is S^1-invariant. The next theorem is joint work with Robert Berman:

Theorem

Assume D is balanced and f is psh and S^1-invariant. Then
1. $\hat{f}$ is psh.

2. $\mathcal{E}(\hat{f}) \leq \mathcal{E}(f)$ holds for all such f if and only if D is an ellipsoid.

I will discuss the proof of the first part.

proof of first part

Recall that $\phi(|z|)$ is psh if and only if

$$\psi(s) := \phi(e^s)$$

is convex. The definition of Schwarz symmetrization gives

$$\sigma_f(r) = |\{\phi(|z|) < r\}| = \phi^{-1}(r)^{2n}$$

if we normalize Lebesgue measure so the the unit ball has volume 1.

$$\psi^{-1}(t) = \log \phi^{-1}(t) = (1/2n) \log \sigma_f(t).$$

ψ is convex if and only if its inverse ψ^{-1} is concave. So we need to prove that

$$-\log \sigma_f(t)$$

is convex.

Let

$$\mathcal{D} := \{(t + is, z); f(z) < t\};$$

a pseudoconvex domain (since f is S^1-invariant). Its slices are

$$D_t = \{z; f(z) < t\}.$$

The diagonal Bergman kernel for D_t at the origin is

$$\frac{1}{|\{f < t\}|}.$$

Hence complex Prekopa implies that $\log \sigma_f(t)$ is convex (subharmonic in $(t + is)$ and independent of s).

□

Openness conjecture

Theorem

Let f be psh in the ball and suppose

$$\int_B e^{-f} < \infty.$$

Then there is $\epsilon > 0$ such that

$$\int_{B/2} e^{-(1+\epsilon)f} < \infty.$$

I will prove this when f is S^1-invariant.

Assume first that $f(z) = \phi(|z|)$ is radial. As before

$$\psi(t) = \phi(e^t)$$

is convex.

$$\int_B e^{-f} = \int_{-\infty}^0 e^{-\psi(t)+2nt} dt.$$

We may as well change ψ to $\psi(-t)$ and get

$$\int_0^{-\infty} e^{-\psi(t)+2nt} dt.$$

Assume $\psi(0) = 0$. Then $\psi(t)/t$ is increasing to a limit, a. The integral converges iff $a > 2n$, which is an open condition.
By Schwarz symmetrization the same thing holds for S^1-invariant functions.

□

Suita conjecture

Let D be a domain in the complex plane, containing the origin. Let $G(z)$ be the Green's function of D with pole at the origin. Then

$$G(z) = \log |z|^2 - h(z),$$

where h is harmonic, with boundary values such that $G = 0$ on the boundary of D. Let $c_D = h(0)$, the Robin constant. The following theorem of Blocki and Guan-Zhou solved an old conjecture of Suita:

Theorem

Let B be the diagonal Bergman kernel for D. Then

$$B(0) \geq \frac{e^{-c_D}}{\pi} =: Suit.$$

Define a domain in $\mathbb{C}^2$

$$\mathcal{D} = \{(t + is), z); z \in D, G(z) < t\}.$$

Let

$$D_t = \{z \in D; G(z) < t\}$$

be its slices, and $B_t(0)$ the diagonal Bergman kernel of D_t at the origin.

Then $t \to \log B_t$ is convex on $(-\infty, 0)$.

When $t = 0$, $D_t = D$. When t is close to $-\infty$, D_t is close to the disc

$$\Delta_t = \{|z|^2 < e^{t+c_D}\}.$$

Hence B_t is asymptotic to

$$\frac{e^{-t-c_d}}{\pi} = e^{-t} Suit.$$

It follows that

$$u(t) := \log B_t + t$$

is convex and bounded on the negative half-axis. Therefore it increases. Hence

$$\log B_0 \geq \lim_{t \to -\infty} u(t) = \log Suit,$$

which gives the theorem.

□

Thanks!