Hénon Maps – Restart

$$f(x,y) = (x^2 + c - ay, x)$$

Notation:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = f^n\begin{pmatrix} x \\ y \end{pmatrix}$$

Recurrence scheme

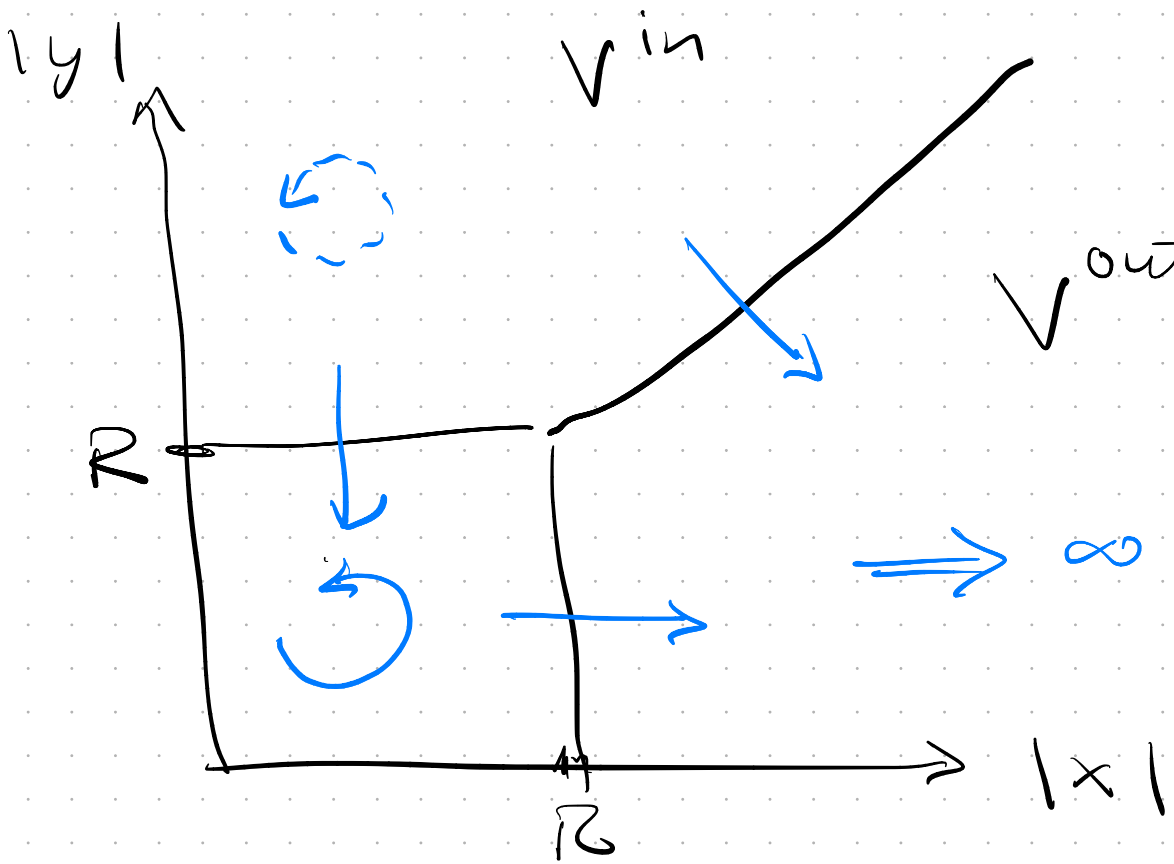
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n^2 + c - ay_n \\ x_n \end{pmatrix}$$

$$x_0 = x$$

$$x_{-1} = y_0 = y$$

$$x_{n+1} = x_n^2 + c - a\,x_{n-1}$$

Behavior in the large: Filtration on $\mathbb{C}^2$



$V^{out} = \{ |x| \geq \max(|y|, R) \}$

$V = \{ |x|, |y| < R \}$

- $K^+ = \bigcap_{n \geq 0} f^{-n} (V^{in} \cup V)$ (orbits bounded)
- $U^+ = \bigcup_{n \geq 0} f^{-n} (V^{out})$ (escape locus)

Exponential escape to infinity in V^{out}

$(x,y) \in V^{out}$ $|x_n| \geq \max(R, |y_n|)$

$$x_{n+1} = x_n \left(x_n + \frac{c}{x_n} - \frac{a y_n}{x} \right)$$

$$|x_{n+1}| \geq |x_n| \left(R - \frac{c}{R} - |a| \right) \geq |x_n| k$$

k for large R

$$\therefore \boxed{|x_n| \geq k^n R}$$

Super-Exponential Escape

$$x_{n+1} = x_n^2 \left(1 + \frac{c}{x_n^2} - \frac{a y_n}{x_n^2} \right)$$

$$|x_{n+1}| \geq |x_n|^2 (1 - \delta)$$

$$\boxed{|x_n| \geq \left((1-\delta) R\right)^{2^n}}$$

Green function on V^{out}

$$G_n^+(x,y) = \frac{1}{2^n}\log|x_n| = \frac{1}{2^n}\log^+|x_n|$$

$$G_{n+1}^+ - G_n^+ = \frac{1}{2^{n+1}}\log|x_{n+1}| - \frac{1}{2^n}\log|x_n|$$

$$= \frac{1}{2^{n+1}}\log\left|\frac{x_{n+1}}{x_n^2}\right| = \frac{1}{2^{n+1}}\log\left|1 + \frac{c-ay_n}{x_n^2}\right|$$

$$= \frac{1}{2^{n+1}}O\left(\frac{1}{|x_n|}\right)$$

Globally:

Thm 1 $\quad G^+(x,y) = \lim_{n\to\infty}\frac{1}{2^n}\log^+|x_n|$

uniformly on compacts

Thm 2

$$\frac{1}{2^n}\log|x_n| \longrightarrow G^+ \quad \text{in } L^1_{loc}(\mathbb{C}^2)$$

- super-exponential growth $\Rightarrow \frac{y_n}{x_n} \longrightarrow 0$ as $n \to \infty$

so definition of G^+ is independent of any norm $\|f^n(x,y)\| = \|(x_n, y_n)\|$

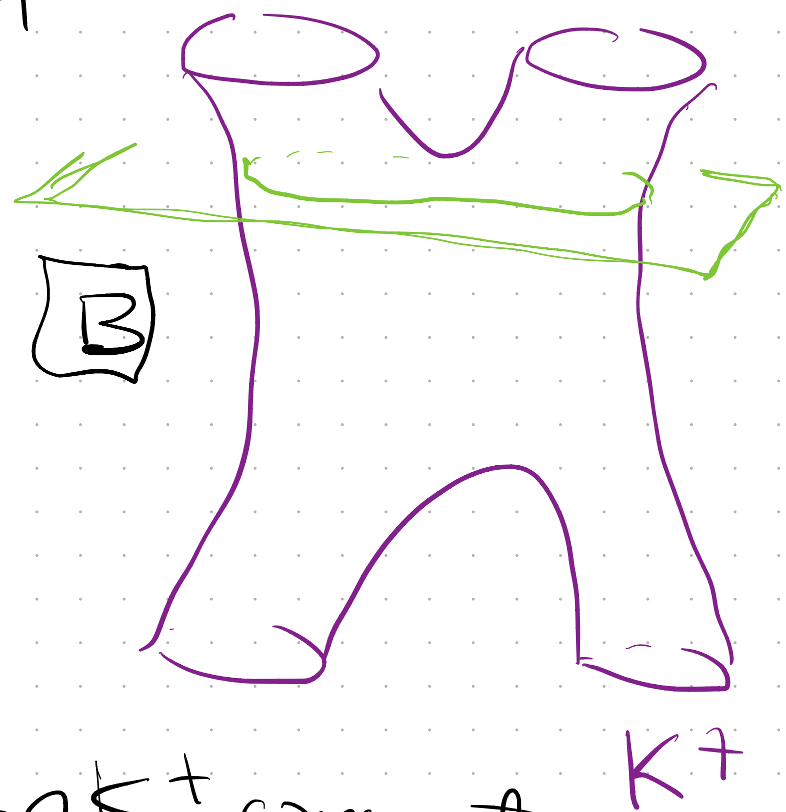
Proof of Theorem 1

Choose open, bounded $B \subset U^+$

By filtration, $\exists\, n_0$

$$f^{n_0}(B) \subset V^{out}$$

$\therefore \frac{1}{2^n} \log^+ \|f^n(x,y)\| \to G^+$ on B



$\therefore$ have convergence on $\mathbb{C}^2 - K^+$

Take horizontal line L $\quad L \cap K^+$ compact

$G^+|_L \longrightarrow 0$ as $(x,y) \to K^+$. Result follows by maximum principle.

Proof of Theorem 2 sequence of psh functions $\frac{1}{2^n}\log|x_n|$ • bounded above

Compactness theorem • $\not\to -\infty$

$\therefore \exists$ convergent subsequence $\xrightarrow[L^1_{loc}]{}$ p.s.h. function g (u.s.c.)

By Theorem 1, we need $g = 0$ a.e. on $\operatorname{int}(K^+)$

(know $g \leq 0$ on $\operatorname{int}(K^+)$)

Let $B \subset \operatorname{int}(K^+)$. Want

$$\frac{1}{2^n}\int_B \log|x_n| \longrightarrow 0$$

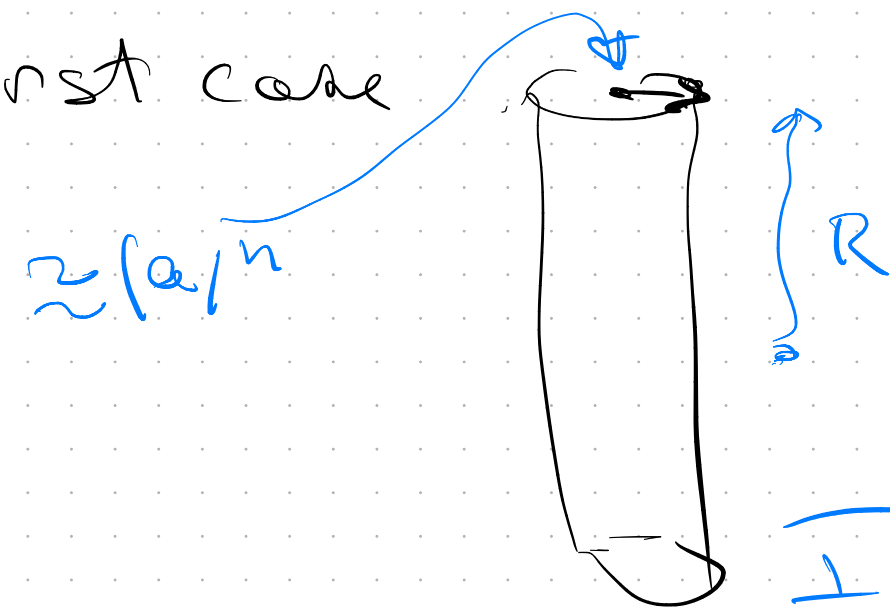
$\operatorname{int}(K^+) \neq \emptyset \Rightarrow$ jacobian $|a| < 1$.

Change of variables formula

$$\int_B \log|x_n| = |a|^{-2n} \int_{f^n(B)} \log|x|$$

- $\mathrm{Vol}\,|f^n(B)| = |a|^{2n}\,\mathrm{Vol}(B)$
- $f^n(B) \subset V = \{ |x|, |y| \le R \}$

Worst case



In this case

$$\int_{|x| < |a|^n} \log|x| \approx |a|^{2n} \log|a|^n$$

$$\frac{1}{2n}\int \approx \frac{1}{2n}\log|a|^n \to 0$$

On $\mathbb{C}^1$: Fundamental solution

$$\Delta \log|z| = c\,\delta_0 \quad \text{(as distributions)}$$

recall $\partial, \bar\partial$ $\quad d = \partial + \bar\partial$,

$$dd^c = 2i\,\partial\bar\partial$$

vector-valued analog of Laplacian

Complex Magic: Poincaré-Lelong Formula

$g(x,y)$ holomorphic function

$$\frac{1}{2\pi}\, dd^c \log|g| = [\{g=0\}]$$

(as currents)

$\pi(x,y) = x$, projection to x-axis

$$x_n = \pi \circ f^n = (f^n)^* \pi$$

$$dd^c \log|x_n| = [\{x_n = 0\}] = (f^n)^* [\{x = 0\}]$$
$$= [f^{-n}(y\text{-axis})]$$

Corollary to Theorem 2:

$$\frac{1}{2^n}[f^{-n}(y\text{-axis})] \longrightarrow \mu^+$$

Thm (B–Smillie) $A = \{p(x,y) = 0\}$

$\exists\, c_A > 0$ such

any algebraic curve

$$\lim_{n \to \infty} \frac{1}{2^n} \log|p(f^n(x,y))| \xrightarrow[L^1_{loc}]{} c\, G^+ \qquad \frac{1}{2^n}[f^{-n}(A)] \to c\,\mu^+$$

Consequences of the fact that

$$\operatorname{Supp} \mu^{\pm} = J^{\pm} = \partial K^{\pm}$$

Corollary 1 : If M is invariant Riemann surface ($fM = M$), then M is not algebraic.

∴ stable manifolds never algebraic

Corollary 2. If $A \subset \mathbb{C}^2$ is algebraic curve, then

$$A \not\subset K^{\pm}$$

$$A \cap J^{\pm} \neq \emptyset.$$

What is the structure of $\mu^\pm$?

If $\mathcal{D} \subset \mathbb{C}^2$ is a complex disk, $dd^c\big|_{\mathcal{D}}$ = "intrinsic Laplacian on $\mathcal{D}$"

slice measure on $\mathcal{D}$

$$\mu^+\big|_{\mathcal{D}} := \left(dd^c\big|_{\mathcal{D}}\right)\left(G^+\big|_{\mathcal{D}}\right)$$

<u>Prop.</u> μ^+ is (defined by) the family of slice measures $\mu^+\big|_{\mathcal{D}}$.

G^+ = Universal Green function for K^+

$K \subset \mathbb{C}$ compact, Green function G_K:

1. harmonic on $\mathbb{C} - K$
2. $G_K = 0$ on K
3. $G_K \sim \log|z|$ at ∞

harmonic measure

$$\mu_K = \Delta G_K \,,$$

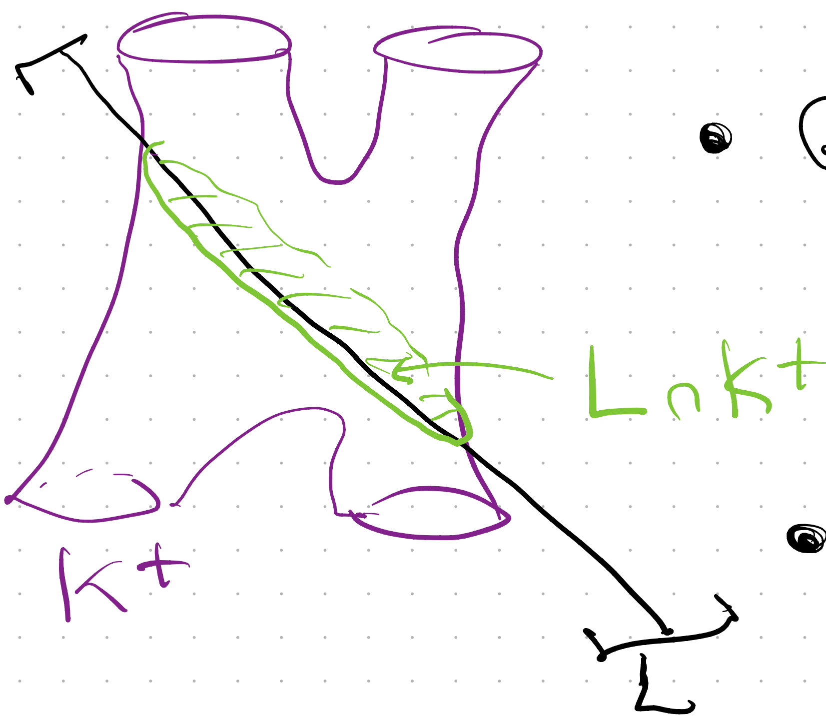
$$\mu^+ = \frac{1}{2\pi}\, dd^c G^+$$

L = complex line $\subset \mathbb{C}^2$

$\cong \mathbb{C}$

$\{ax + by = c\}$

$a \neq 0$



- $G^+\big|_L$ = Green function $G_{K^+ \cap L}$ inside L

(if $a = 0$, $G^+\big|_L = \frac{1}{2}$ Green function)

- $\mu^+\big|_L$ = harmonic measure Ω slice

What is the nature of $\mu^\pm = dd^c G^+$

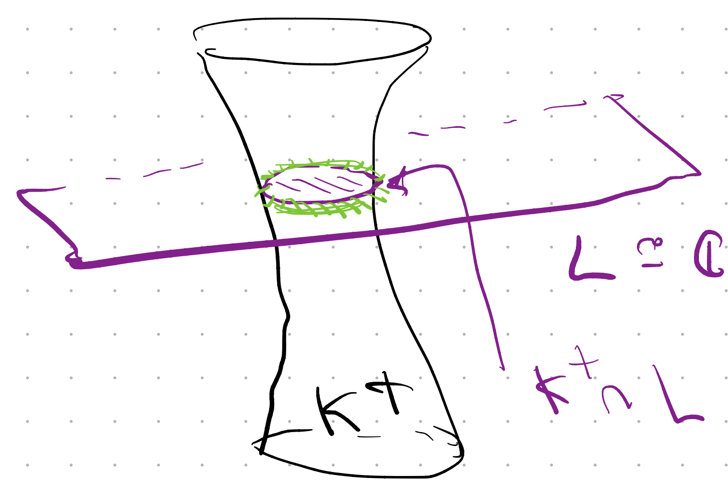
- If $D \subset \mathbb{C}^2$ is a complex disk, you can take slice measure

$$\mu^\pm\Big|_D = dd^c\left(G^\pm\Big|_D\right)$$

(measure)

$\mathbb{C}$
$\cong$
If $L \subset \mathbb{C}^2$ is a complex line, then $G^+\Big|_L$ is the classical Green function of $K^+ \cap L$ inside L

$$\mu^+\Big|_L = \text{harmonic measure}$$ (green)



Universal harmonic measure for K^+

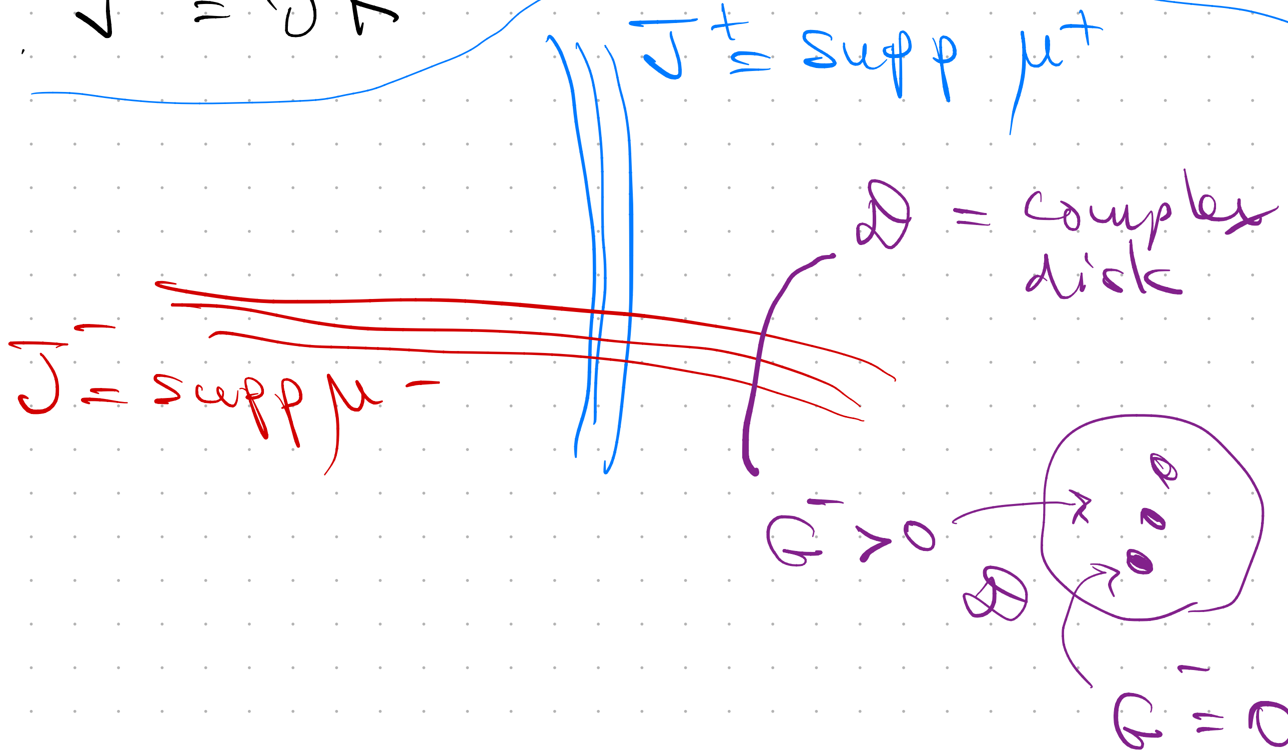
Complex Hénon Maps: $f(x,y) = (x^2+c-ay,\ x)$

$(x_n, y_n) = f^n(x,y)$ $\qquad G^+ = \lim\limits_{n\to\infty} \frac{1}{2^n}\log|x_n|$

$U^+ = \{G^+ > 0\}$ $\qquad G^- = \lim\limits_{n\to\infty} \frac{1}{2^n}\log|x_{-n}|$

$K^+ = \{G^+ = 0\}$ $\qquad$ Currents $\mu^\pm = dd^c G^\pm$

$J^+ = \partial K^+$ $\qquad$ "vector Laplacian"



nontrivial intersection

$\{G^- = 0\} \cap \mathcal{D} \neq \emptyset$

$\{G^- > 0\} \cap \mathcal{D} \neq \emptyset$

- nontrivial intersection $\Rightarrow c = \text{mass}\left(\mu^-|_{\mathcal{D}}\right) > 0$

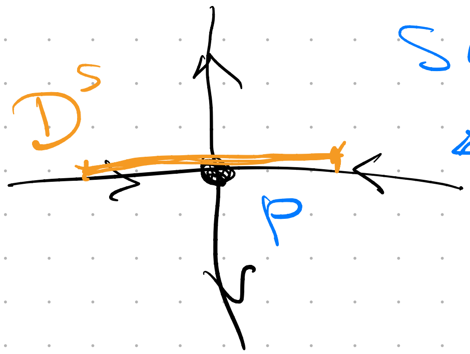
μ^+ represents the limit dynamics of f acting on complex disks

Thm (B-Smillie) If D is a complex disk and $c = \text{mass}(\mu^-|_D)$ and $\mu^-|_{\partial D} = 0$,* then

$$\frac{1}{d^n}(f^n)^*[D] = \frac{1}{d^n}[f^{-n}D] \to c\,\mu^+$$

* no big deal (explain on blackboard)

1. Consequence: Stable manifolds are dense in J^+.



saddle point p

stable manifold $W^s(p)$

Thm (B-Smillie) If D^s is a stable disk containing p, then

- $\frac{1}{2^n}[f^{-n}D^s] \longrightarrow c\,\mu^+$, $c>0$.

- $W^s(p) = \bigcup_{n>0} f^{-n}(D^s)$ dense in $J^+ = \operatorname{supp}\mu^+$

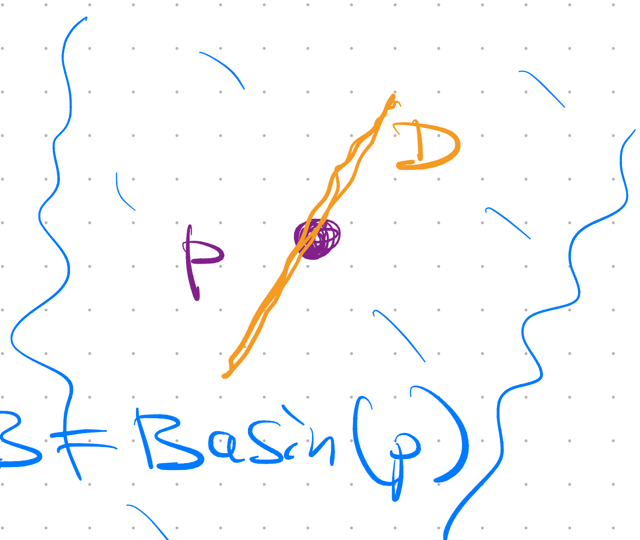
hyperbolic [picture]

2. Consequences for Basins of Attraction

- If p = attracting (periodic) point then $p \in J^-$, (since $Df(p)$ repelling)

$\therefore p \in \operatorname{supp} \mu^-$

$\therefore \exists\ D$ = complex disk $\mu^-|_D \neq 0$



$B = \text{Basin}(p)$

$$\frac{1}{2^n}\left[f^{-n} D\right] \longrightarrow c\,\mu^+$$

$$\operatorname{supp} \mu^+ \subset \overline{B}$$

$\therefore \operatorname{supp} \mu^+ = \partial B = J^+$

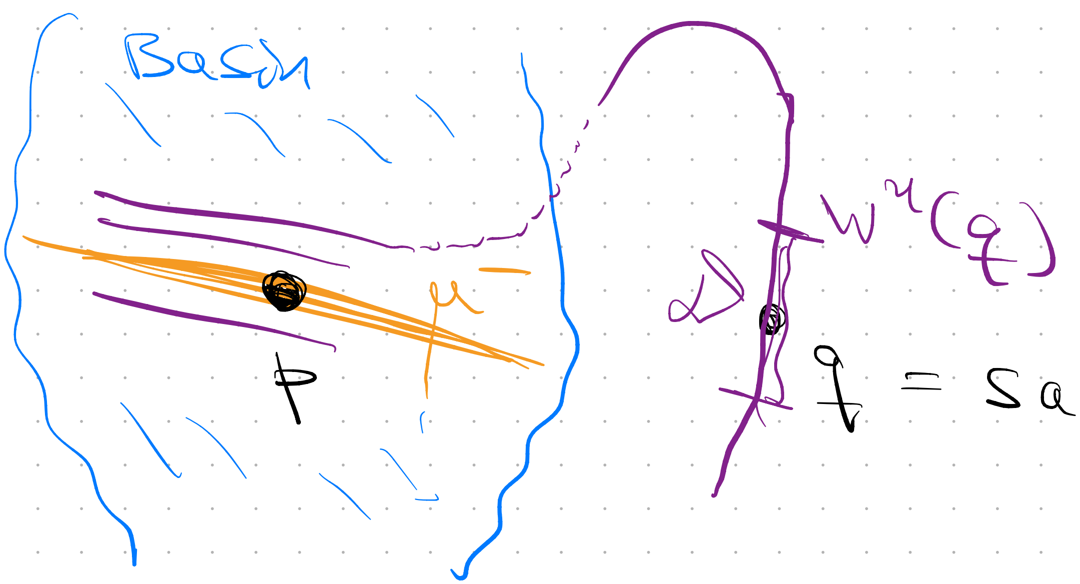
Corollary: All basins have the same boundary.

[And it is possible to have ∞-many basins.]

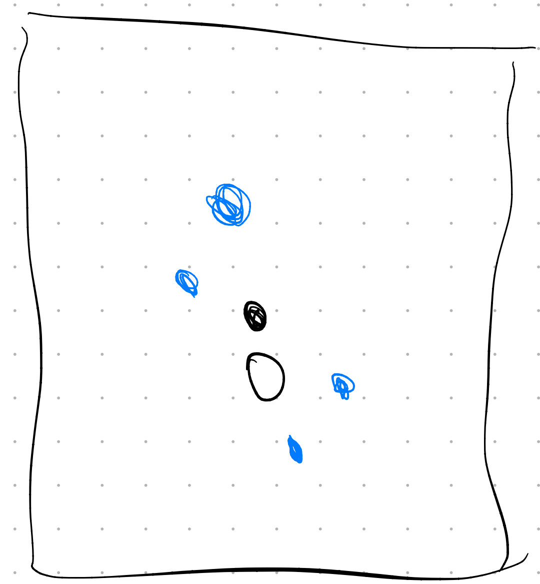
3. Does a map f have a basin?

How can you search for a basin?

- All Basins "visible" in every unstable slice picture.



q = saddle



$f^n \mathcal{D}$ must accumulate near p in order

$\frac{1}{2^u}\left[f^n \mathcal{D}\right] \to c\,\mu^-$. $\therefore$ $W^u(q)$ enters $\mathcal{B}$

Dynamics of a Polynomial $p : \mathbb{C} \circlearrowleft$

Böttcher coordinate $\varphi : \{|z| > R\} \longrightarrow \mathbb{C}$

$$\varphi \circ p = \varphi^2$$

model 1 J connected (equiv. K connected)

φ extends to conformal equivalence / conjugacy

$$\begin{tikzcd} \mathbb{C} - K \arrow[r, "\varphi"] \arrow[loop below, "p"] & \mathbb{C} - \bar{\Delta} \arrow[loop below, "\sigma"] \end{tikzcd}$$

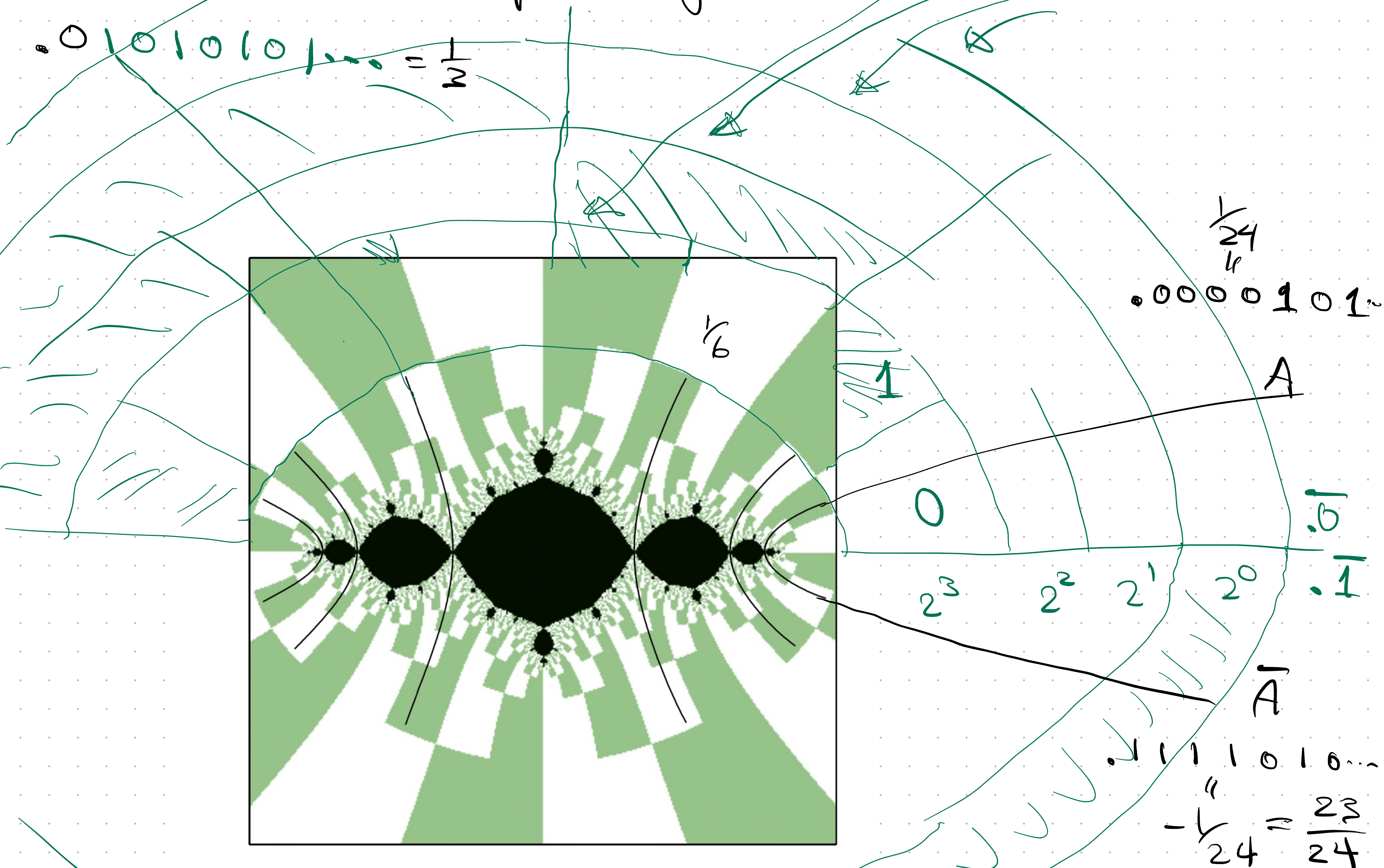
$\sigma(w) = w^2$

$G = \log|\varphi| =$ Green function $\qquad \log|w| =$ Green function

Gradient lines $\longleftrightarrow$ rays

$\{re^{i\theta} : 1 < r < \infty\}$

model 1 in computer picture $\quad G = c\,2^{-n}$



Level sets/gradient lines for $G = \log|\varphi|$

model 2 $\quad p : \mathbb{C} \longrightarrow \mathbb{C}$

J connected

If p is expanding (or if J is locally connected), then

$$\mathbb{C} - K \xleftarrow{\psi := \varphi^{-1}} \mathbb{C} - \bar{\Delta}$$

extends continuously to $\partial\Delta$.

This gives a quotient map

$$\begin{array}{ccc} \psi : S^1 & \longrightarrow & J \\ \circlearrowleft & & \circlearrowleft \\ \sigma & & p \end{array}$$

In computer picture, gradient lines are coded according to Böttcher $\varphi(z) = .b_1 b_2 b_3 \cdots (2)$

Solenoid

$$\Sigma_0 = \{ \underline{s} = (s_i)_{i \in \mathbb{Z}} : |s_i| = 1, \; s_{i+1} = s_i^2 \}$$

Σ_0 is a **compact group**

$$= \text{proj lim} \left(S^1 \xrightarrow{\sigma} S^1 \right)$$

$$\sigma(s) = s^2$$

$X_2 = \{0,1\}^{\mathbb{Z}}$ = bilateral shift space

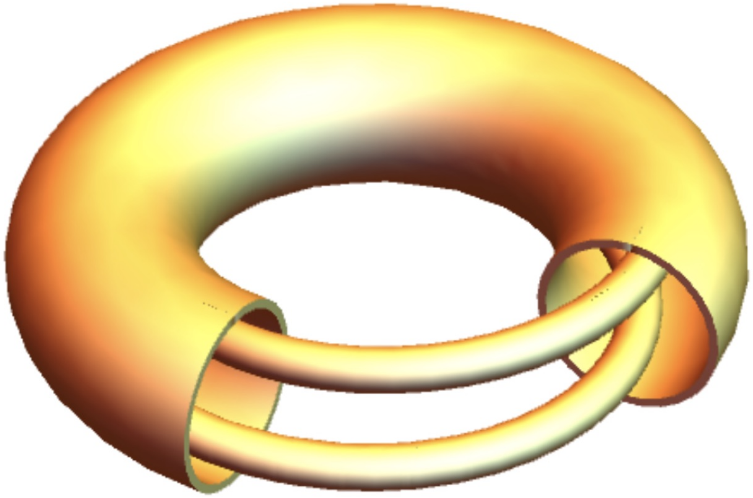
$$\Sigma_0 \cong X_2 / \sim \qquad (w\, 0\, \bar{1}) \sim (w\, 1\, \bar{0})$$

j $\quad$ j

word infinite to left

"same number in binary digits"

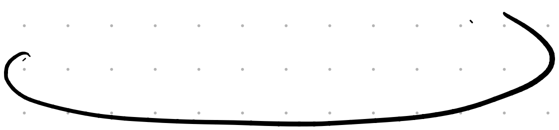
$$\boxed{\sigma : \Sigma_0 \to \Sigma_0 \quad \text{homeo}}$$

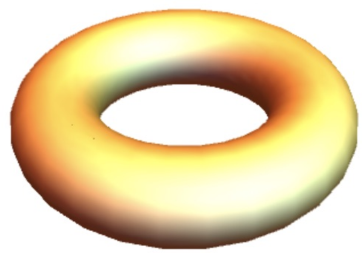


$$\tau : M = S^1 \times \text{Disk} \circlearrowleft$$

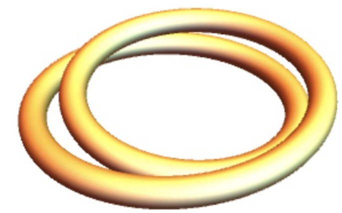
$$\Sigma_0 = \bigcap_{n \geq 0} \tau^n(M)$$

$\pi \downarrow$

S^1 



M



$f(M)$

$f^2(M)$

$f^3(M)$

$\pi : M \longrightarrow S^1$ is a "covering space" with Cantor set as fiber.

Σ_0 is a compact group

identity is $\underline{1} = (\cdots 1, 1, 1, \cdots)$

$\underline{E}(t) = \left(e^{2\pi i 2^n t}\right)_{n \in \mathbb{Z}}$. $\quad \underline{1} = \underline{E}(0)$

$\underline{E}(t_1 + t_2) = \underline{E}(t_1) \cdot \underline{E}(t_2)$

path component of $\underline{1}$ is $\{ \underline{E}(t) : t \in \mathbb{R} \}$

path component of $\underline{s}$ is

$\{ \underline{s} \cdot \underline{E}(t) : t \in \mathbb{R} \}$

Identify $\underline{s}$ with the digital representation

$\underline{s} = (\cdots s_{-1}\ s_0\ s_1 \cdots)$ $\quad s_0 = e^{2\pi i \theta_0}$

$\theta_0 = .b_1 b_2 b_3 \cdots$ (binary)

$b_0 =$ choice of $\pm\sqrt{s_0}$

$b_{-1} =$ " " $\pm\sqrt{s_{-1}}$

Historical digression

Hubbard: Solenoids are everywhere in complex Hénon maps.

Recall: $G^+ : \mathcal{U}^+ \longrightarrow (0, \infty)$

Topology of level sets:

Theorem [HOV 1]

$$\{G^+ = c\} \underset{\text{homeo}}{\cong} S^3 - \Sigma_0$$

real torus $S^1 \leftarrow$ algebraic torus $\mathbb{C}^*$

Complex solenoid

$$\Sigma_* = \varprojlim \left(\sigma(z) = z^2 : \mathbb{C}^* \circlearrowleft \right)$$

$$\left\{ \underline{z} = (z_j)_{j \in \mathbb{Z}} : \quad z_{j+1} = z_j^2 \right\}$$

group structure $\quad \underline{z} \cdot \underline{w} = (z_j w_j)_{j \in \mathbb{Z}}$

$\sigma(\underline{z}) = \underline{z}^2 : \Sigma_* \circlearrowleft$ is homeomorphism

$\pi : \Sigma_* \longrightarrow \mathbb{C}_* \qquad$ polar coordinates

$\pi(\underline{z}) = z_0$

This is essentially a real solenoid: $\quad \Sigma_* \cong (0,\infty) \times \Sigma_0 \qquad \underline{z} = r \cdot \underline{s}$

Complex solenoid over $\mathbb{C} - \bar{\Delta}$

$$\Sigma_+ := \varprojlim \left(\mathbb{C} - \bar{\Delta},\ \sigma(w) = w^2 \right)$$

$$= \left\{ \underline{\xi} = (\dots, \xi_{-1}, \xi_0, \xi_1, \dots) : \xi_{j+1} = \xi_j^2,\ \xi_i \in \mathbb{C} - \bar{\Delta} \right\}$$

$$\pi : \Sigma_+ \longrightarrow \mathbb{C} - \bar{\Delta}$$

$$\pi(\underline{\xi}) = \xi_0$$

Σ_+ has "complex structure"

[locally homeo to (complex disk) $\times$ Cantor set]

semigroup $\quad \underline{\xi} \cdot \underline{\eta} = (\xi_j \eta_j)_{j \in \mathbb{Z}}$

$$\sigma : \Sigma_+ \longrightarrow \Sigma_+ \qquad \sigma(\underline{\xi}) = \underline{\xi}^2 = \text{shift map}$$

(homeo)

$$(\sigma(\underline{\xi}))_j = \xi_{j+1}$$

fixed point $\underline{0} = (\ldots 0\,0\,0 \ldots) = (\ldots 1\,1\,1 \ldots) \in \Sigma_0$

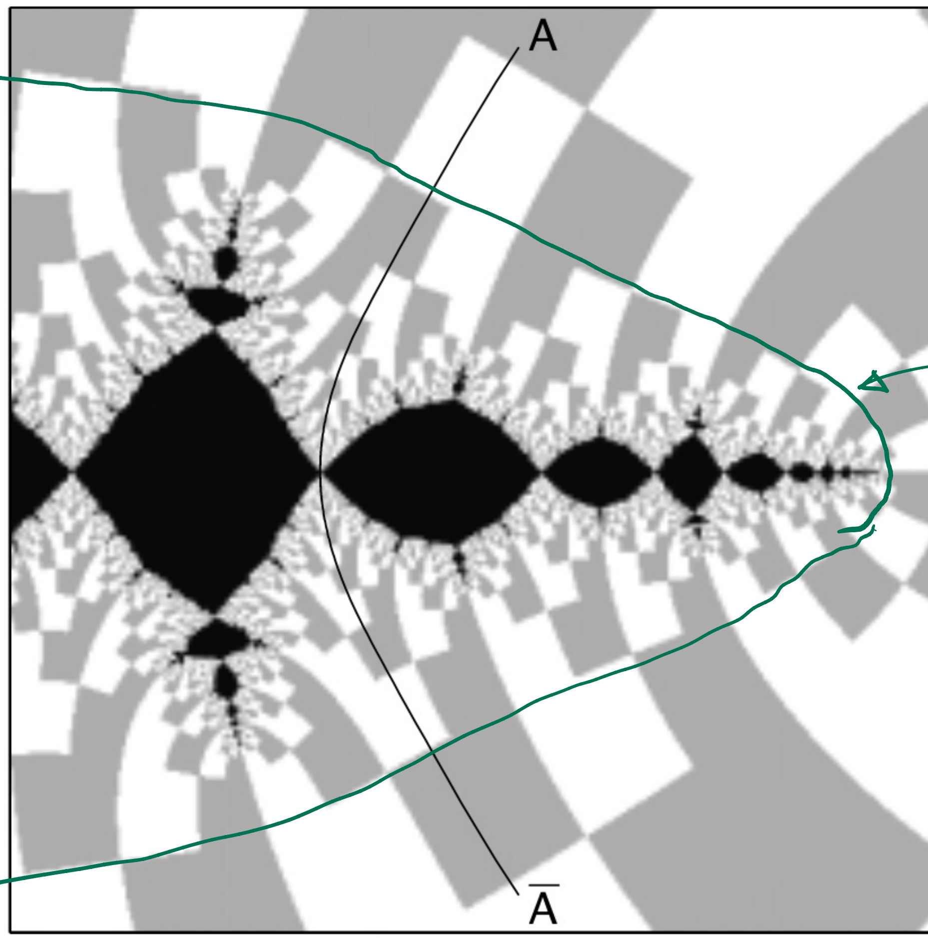
$W^u(\underline{0}) = \{ \underline{s} = ({}^\infty 0\, 1\, w) = ({}^\infty 1\, 0\, w) \}$

Unstable slice picture for perturbation of Basilica.

solenoidal coordinates

- picture is invariant under $\zeta \to \lambda \zeta$

Level set $\simeq \mathbb{R}$



Level sets / gradient lines for $G^+|_{W^u(p)}$

$G^+ = C$

Can read solenoidal coding of gradient lines

$A = \ldots (10)^\infty$
$= {}^\infty 0 \ldots (10)^\infty$

R. Oliva

Our goal: Give analogues of 1-D Models

$$J_+^- := J^- - K^+$$

Model 1 for Hénon maps:

$$\text{“} (J_+^-, f) \cong (\Sigma_+, \sigma) \text{”}$$

Model 2 for Hénon maps:

"show that there are external rays, and these rays land at J"

"There is a quotient map

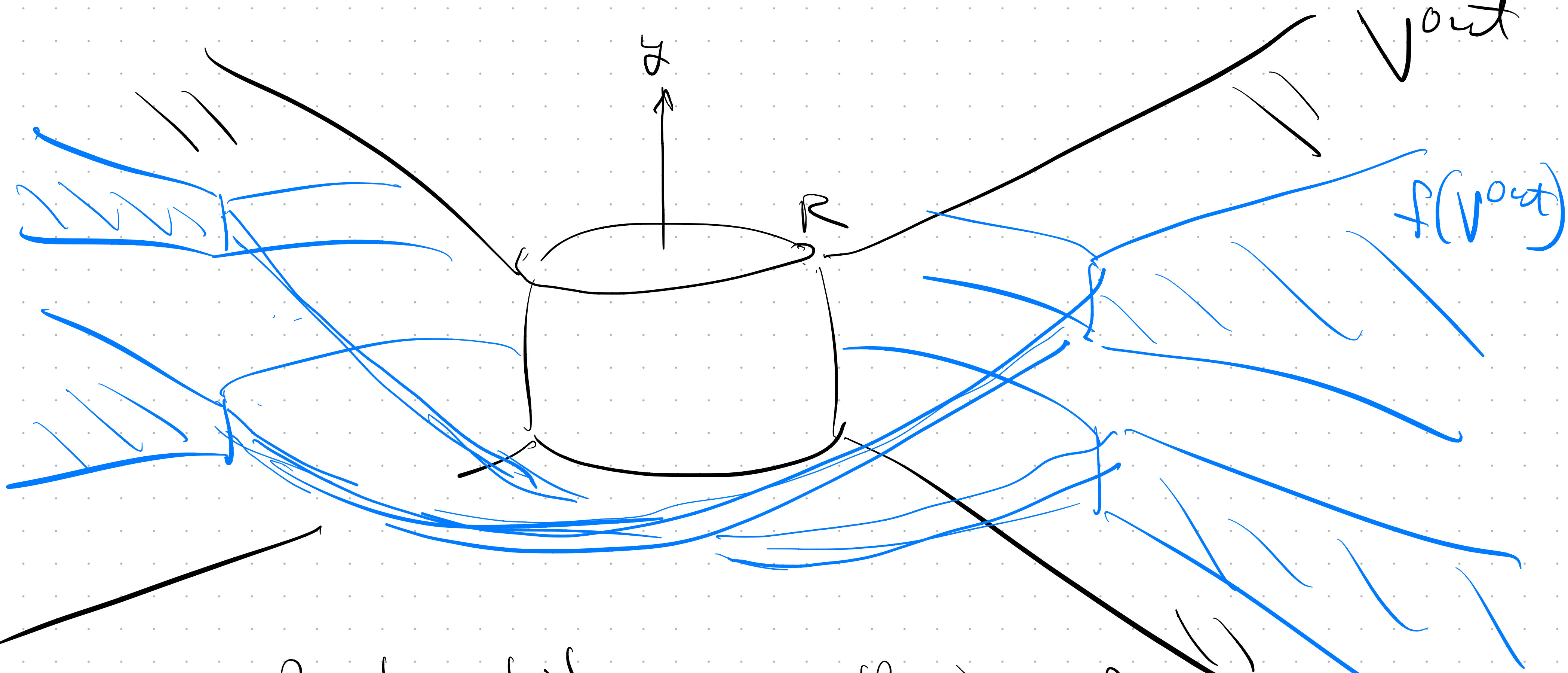
$$\eta : (\Sigma_0, \sigma) \longrightarrow (J, f)$$

with finite fibers"

Historical Interlude

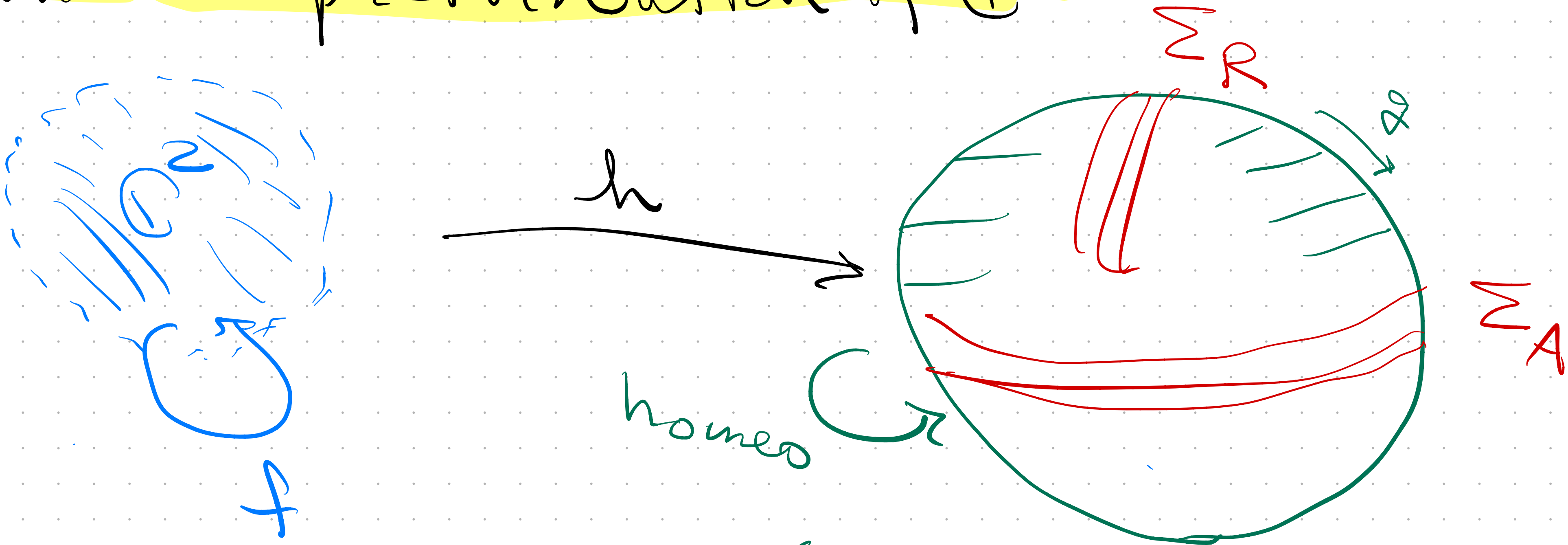
The "solenoidal model" proposed by Hubbard.

recall $V^{out} = \{ R \leq |x|, |y| \leq x \}$



This looks like something that should give a "solenoid at infinity"!

Want compactification of $\mathbb{C}^2$



X = closed 4-ball

$h : \mathbb{C}^2 \xrightarrow{\text{diffeo}} \operatorname{int}(X)$

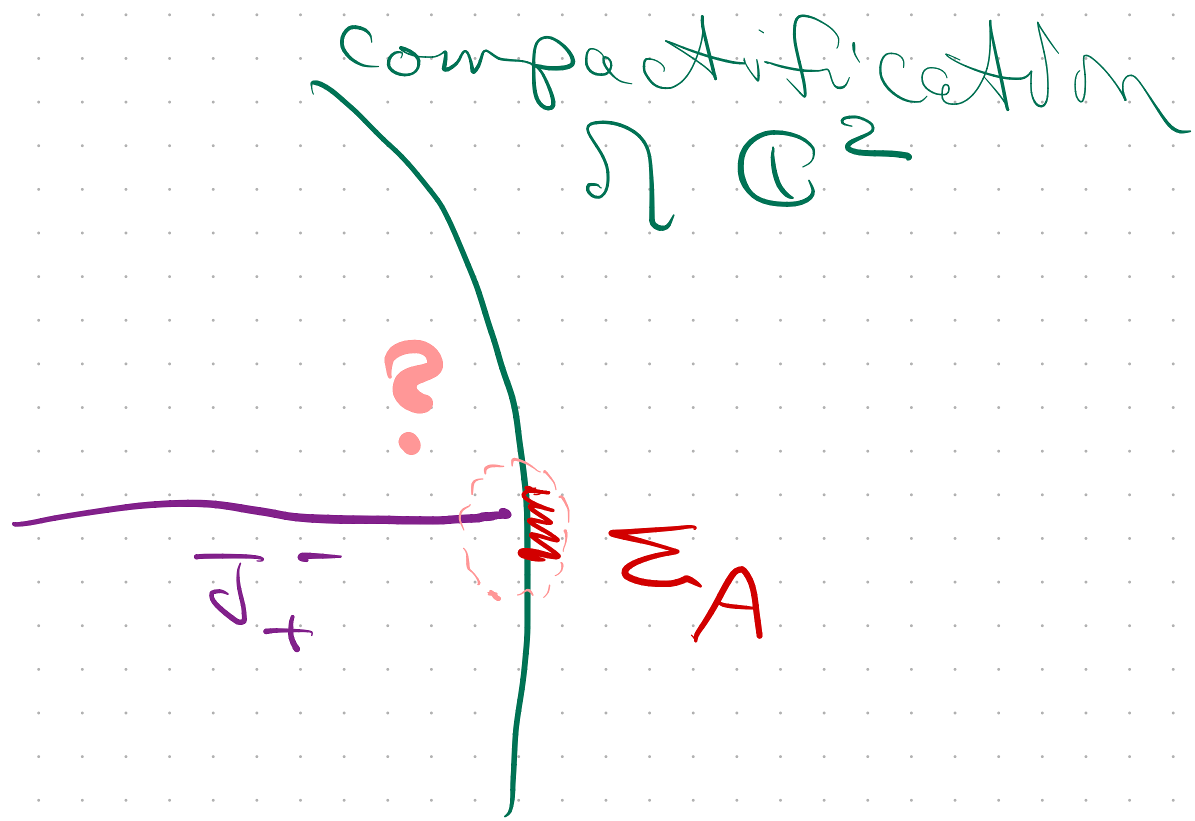
$\Sigma_{A/R}$ = attracting/repelling solenoids for $g|\partial X$

formulated by Hubbard & Oberste-Vorth 1994

proof given by Buzzard 2000.

Problem / Question

We will show (in some cases) J_+^- is a solenoid. What is the relationship between J_+^- and Σ_A?



Back to our topic:

$$f(x,y) = (x^2 + c - ay,\ x)$$

$$(x_n, y_n) = f^n(x,y)$$

$$x_{n+1} = x_n^2 + c - ay_n = x_n^2\left(1 + \frac{c - ay_n}{x_n^2}\right)$$

If $(x,y) \in V^{out}$ then $(x_n, y_n) \in V^{out}$

$$|x_n| \geq |y_n| \geq k^n$$

may take roots

$$O\left(\frac{1}{k^n}\right)$$

$$x_{n+1}^{\frac{1}{2}} = x_n\left(1 + \frac{c - ay_n}{x_n^2}\right)^{\frac{1}{2}}$$

$$x_{n+1}^{\frac{1}{4}} = x_{n-1}\left(1 + \frac{c - ay_{n-1}}{x_{n-1}^2}\right)^{\frac{1}{2}}\left(1 + \frac{c - ay_n}{x_n^2}\right)^{\frac{1}{4}}$$

...

$$x_{n+1}^{\frac{1}{2^n}} = x_0\left(1 + \frac{c - ay_0}{x_0^2}\right)^{\frac{1}{2}} \cdots \left(1 + \frac{c - y_n}{x_n^2}\right)^{\frac{1}{2^n}}$$

Theorem [HOV 1]

The product

$$\varphi^+ := x \prod_{n=0}^{\infty} \left(1 - \frac{c - a y_n}{x_n^2}\right)^{\frac{1}{2^n}}$$

converges uniformly on V^{out},

$$\varphi^+ \circ f = (\varphi^+)^2$$

Have Böttcher coordinate on V^{out}

First step to Model 1

Theorem [BS 6] Suppose J is connected. then

- there is a neighborhood $\mathcal{U}$ of J_+^-, and φ^+ extends holomorphically to $\tilde{\varphi}^+ : \mathcal{U} \longrightarrow \mathbb{C}$
- For $p \in J_+^-$,

$$\Phi^+(p) = \left(\tilde{\varphi}^+(f^n(p)) \right)_{n \in \mathbb{Z}}$$

defines a semi-conjugacy

$$\Phi^+ : (J_+^-, f) \longrightarrow (\Sigma_+, \sigma)$$

Next time flesh out Models 1 and 2

- Criteria to determine whether J is connected
- proof of the extension $\tilde{\varphi}^+$
- how to replace Φ^+ with a conjugacy?
- definition of external rays
- prove that external rays land at J