Burns-Krantz rigidity in non-smooth domains

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The above theorem is sharp: $F(\lambda) = \lambda - \frac{(\lambda-1)^3}{10}$, $\lambda \in \mathbb{D}$, maps $\mathbb{D}$ to $\mathbb{D}$.

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General problem is to study whether the pair (D, p), where $p \in \partial D$ satisfies *the Burns-Krantz rigidity property (of order $m \geq 3$)*.

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Further discussion

Consider the domain $D := \{z \in \mathbb{D}_* \times \mathbb{C} : |z_2| < \exp(-1/|z_1|^2)\}$. Then $(D, 0)$ does not satisfy the Burns-Krantz rigidity property.

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Below we show that the Burns-Krantz rigidity property holds for non-smooth boundary points.

Theorem

The pair $(\mathbb{D}^n, p)$, $p \in \partial\mathbb{D}^n$, satisfies the Burns-Krantz rigidity property.

A tour to the Lempert theory

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In the case of strongly convex domains complex geodesics extend regularly to the boundary. And the left inverses may be chosen to be regular, too. Additionally one may construct left inverses so that they have absolute values one only on the boundary of complex geodesics.

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We show below the invariance of left inverses under Burns-Krantz mappings.

Invariance of left inverse under Burns-Krantz mapping

Proposition

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It is sufficient to apply the Burns-Krantz rigidity theorem for the disc to get the conclusion. □

Proof of the Burns-Krantz property for the bidisc

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By Proposition $F_1(\lambda, a(\lambda)) = \lambda$ for all such a's so $F_1(z) = z_1$.

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Analoguously we get that $F_2(z) = z_2$.

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Additionally, they do not touch the boundary of the royal variety.

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- Can one apply the method to more (preferably non-smooth) domains?