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A new approach to the study of $m-$convex functions.

A. Sadullaev

National University of Uzbekistan

sadullaev@mail.ru

In this work we will give a new approach to the study of $m-$convex $(m-cv)$functions in the domain $D\subset\mathbb{R}_x^n$, $(1\le m\le n)$.

1. If the potential theory in the class of $m-$subharmonic functions is based on differential forms and currents $\left(dd^c u\right)^k\wedge\beta^{n-k}\ge 0,\ k=1,2,\dots,n-m+1,$ where $\beta=dd^c\|z\|^2$ the standard volume form in $\mathbb{C}^n$, then the potential theory in the class of $m-cv$ functions is based on Borel measures of a completely different nature, namely, on Hessians $H^k(u)\ge 0,\ k=1,2,\dots,n-m+1.$

Remember, for doubly smooth function $u(x)\in C^2(D)$, in the domain $D\subset\mathbb{R}^n$, the orthogonal matrix $\left(\frac{\partial^2 u}{\partial x_j\partial x_k}\right)$ after a suitable orthonormal transformation can be converted to diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j\partial x_k}\right)\to\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \lambda_n\end{pmatrix},$$

where $\lambda_j=\lambda_j(x)\in\mathbb{R}-$are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j\partial x_k}\right)$. Let $H^k(u)=H^k(\lambda)=\sum_{1\le j_1<\dots<j_k\le n}\lambda_{j_1}\dots\lambda_{j_k}$ be the Hessian of degree k of the eigenvalue vector $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)$.

Definition 1. *A twice smooth function* $u(x)\in C^2(D)$ *is called* $m-$*convex in* $D\subset\mathbb{R}^n$, $u\in m-cv(D)$, *if its eigenvalue vector* $\lambda=\lambda(x)=(\lambda_1(x),\lambda_2(x),...,\lambda_n(x))$ *satisfies the conditions*

$$m-cv\cap C^2(D)=\{H^k(u)=H^k(\lambda(x))\geq 0,\ \forall x\in D,\ k=1,...,n-m+1\}.$$

When $m=n$ the class $n-cv$ coincides with the class of subharmonic functions $sh=\{\lambda_1+\lambda_2+...+\lambda_n\geq 0\}$, and when $m=1$ it coincides with the class of convex functions $cv=\{\lambda_1\geq 0,\lambda_2\geq 0,...,\lambda_n\geq 0\}$. Moreover, $cv=1-cv\subset 2-cv\subset...\subset n-cv=sh.$ The theory of subharmonic functions is developed and important part of the Theory of Functions and Mathematical Physics and the Theory of Convex Functions is well studied and reflected in the works of A. Aleksandrov, I. Bakelman, A. Pozdnyak, V. Pogorelov, etc. (see [Al1, Al2, B1, B2, Po]). For $m>1$ this class was studied in a series of works by N. Ivochkina, N. Trudinger, H. Wong, S.Y. Lee et al. (see [TW1-TW3], [CW], [ITW]).

If we want to construct a good theory of $m-cv$ functions, then the class of functions $C^2(D)$ is not enough. For example, if we want to solve the equation (the Dirichlet problem)

$$H^{n-m+1}(u)=f(u,x),$$
$$u|_{\partial D}=\varphi$$

or want to work with extremal $m-cv$ functions, we will need to extend the class $C^2(D)$ to the class of upper semi-continuous functions.

In the work of N. Trudinger, H. Wong [TW3], $m-cv$ functions are introduced in the class of upper semi-continuous functions $u(x)$ in the domain $D\subset\mathbb{R}^n$, using the so-called, viscosity definition: that $H^k(q)\geq 0,\ k=1,2,...,n-m+1,$ for any quadratic polynomial $q(x)$ such that the difference $u(x)-q(x)$ has only a finite number of local maximums in the domain

D. In addition, in this work, the maximum degree operator $H^{n-m+1}(u)$, was defined as a Borel measure.

We give a new approach to the study of $m-cv$ functions, establishing their connection with $m-$subharmonic (sh_m) functions in complex space $\mathbb{C}^n$.

The theory of sh_m-functions is based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geq 0,\ k=1,2,...,n-m+1,$ where $\beta = dd^c \|z\|^2 -$ the standard volume form in $\mathbb{C}^n$. Theory of sh_m-function is well developed, currently is the subject of study by many mathematicians (Z. Blocki [Bl], S .Dinev and S.Kolodziej [DK], S. Li [Li], H.Ch.Lu [Lu1, Lu2], etc.). A fairly complete review of this theory is available in the survey article by A. S. and B. Abdullaev [AS] in the Proceedings of MIRAN.

A doubly smooth function $u(z)\in C^2(D),\ D\subset\mathbb{C}^n$, *is called (strongly)* $m-$*subharmonic,* $u\in sh_m(D)$, *if at each point of the domain* D

$$\begin{gathered} sh_m(D)=\left\{\ u\in C^2 : (dd^c u)^k \wedge \beta^{n-k} \geq 0,\ k=1,2,...,n-m+1\right\} = \\ =\left\{\ u\in C^2 : dd^c u \wedge \beta^{n-1} \geq 0, (dd^c u)^2 \wedge \beta^{n-2} \geq 0,..., (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\right\}, \end{gathered} \tag{1}$$

where $\beta = dd^c \|z\|^2$ *is the standard volume form in* $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ are closely related to Hessians. For a doubly smooth function $u\in C^2(D)$, the second order differential form $dd^c u = \frac{i}{2}\sum_{j,k}\frac{\partial^2 u}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k$ (at a fixed point $o\in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\bar z_1 + ... + \lambda_n dz_n \wedge d\bar z_n\right]$, where $\lambda_1,...,\lambda_n$ are eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar z_k}\right)$, which are real: $\lambda = (\lambda_1,...,\lambda_n)\in\mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. It is easy to see that

$$\left(dd^c u\right)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u)\beta^n, \tag{2}$$

where $H^k(u) = \sum_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of the degree k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Consequently, a doubly smooth function $u(z) \in C^2(D),\ D \subset \mathbb{C}^n$, is $m-$subharmonic, if at each point $o \in D$ the inequalities hold

$$H^k(u) = H^k_o(u) \ge 0,\ k = 1, 2, \ldots, n-m+1. \tag{3}$$

Note that the concept of a $m-$subharmonic function in a generalized sense is also defined in the general case for upper semicontinuous functions.

Definition 2. *A function* $u(z)$,*defined in a domain* $D \subset \mathbb{C}^n$ *is called* sh_m, *if it is upper semi-continuous and for any doubly smooth* sh_m *functions* $v_1, \ldots, v_{n-m} \in C^2(D) \cap sh_m(D)$ *the current* $dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ *defined as*

$$\begin{aligned} &\left[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}\right](\omega) = \\ &= \int u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0} \end{aligned} \tag{4}$$

is positive,

$$\int u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \ge 0, \quad \forall \omega \in F^{0,0}, \omega \ge 0.$$

Here $F^{0,0}(D)$ is a family of infinitely smooth, finitely supported in D functions.

In Blocki work [Bl] it was proven that this definition is correct, that for functions $u \in C^2(D)$ this definition coincides with the original definition of sh_m-functions. Moreover, in the class of bounded sh_m-functions, operators $\left(dd^c u\right)^k \wedge \beta^{n-k} \ge 0,\ k = 1, 2, \ldots, n-m+1$ are defined as Borel measures in a domain D (see [Bl], [AS]).

2. In this work, we propose a completely different approach to the study of $m-cv$ functions, based on relationships $m-cv$ functions and sh_m-functions, using a rich and well-studied properties of sh_m-functions. To do this, we embed a real space $\mathbb{R}_x^n$ into a complex space $\mathbb{C}_z^n$, $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n \left(z = x + iy\right)$, as a real $n-$dimensional subspace. Then, we lift the function $u\left(x\right)$, given in the domain $D \subset \mathbb{R}_x^n$ to the domain $\Omega = D \times i\mathbb{R}_y^n \subset \mathbb{C}_z^n$, assuming it a constant on parallel planes $\Pi_{x^0} = \left\{z \in \mathbb{C}^n : x = x^0,\ y \in \mathbb{R}^n\right\}$, $u^c\left(z\right) = u^c\left(x + iy\right) = u\left(x\right)$.

The key result of the work is

Theorem 1. *A twice smooth function* $u\left(x\right) \in C^2\left(D\right)$, $D \subset \mathbb{R}_x^n$, *is* $m-cv$ *in* D, *if and only if function* $u^c\left(z\right) = u^c\left(x + iy\right) = u\left(x\right)$, *that does not depend on variables* $y \in \mathbb{R}_y^n$, *is* sh_m *in domain* $D \times i\mathbb{R}_y^n$.

To study a $m-$convex function $u\left(x\right)$, we extend it into complex space $\mathbb{C}^n$ as sh_m-function $u^c\left(z\right)$, and then apply the known properties of $u^c\left(z\right) \in sh_m$ to $u\left(x\right)$, obtain similar properties of convex $m-$function. All the basic properties of $m-$convex functions were obtained in this way by me and my students, using connections $m-cv$ and sh_m- functions.

As a result, we significantly complement the previously available results in $m-cv$ function theory and obtain a number of new results. In particular, we give a complete construction of the Potential Theory in the class of $m-cv$ functions.

Theorem 1 allows us to define a $m-$convex function in the class of upper semicontinuous functions.

Definition 3. *An upper semi-continuous function* $u\left(x\right)$ *in a domain* $D \subset \mathbb{R}_x^n$ *is called* $m-$*convex, if the function* $u^c\left(z\right)$ *is strongly* $m-$ *subharmonic,* $u^c\left(z\right) \in sh_m\left(D \times i\mathbb{R}_y^n\right)$.

3. The definition 3 is convenient in the study of $m-$convex functions, transferring known properties of sh_m-functions to the class $m-cv$. We present some non-trivial ones

-- (Approximation). We take the standard kernel $K_\delta(x)=\frac{1}{\delta^n}K\left(\frac{x}{\delta}\right)$, $\delta>0$, where

-- $K(x)=K(|x|)$;

-- $K(x)\in C^\infty(\mathbb{R}^n)$;

-- support $\operatorname{supp}K=B(0,1)$;

-- $\int\limits_{\mathbb{R}^n}K(x)dx=\int\limits_{B(0,1)}K(x)dx=1.$

Then the convolution

$$u_\delta(y)=\int\limits_D u(x)K_\delta(x-y)dx=\int\limits_{\mathbb{R}^n}u(x+y)K_\delta(x)dx \tag{5}$$

has the property that $u_\delta(x)\in m-cv(D_\delta)\cap C^\infty(D_\delta)$, where $D_\delta=\{x\in D:\ dist(x,\partial D)>\delta\}$, and which, converges pointwise to the function $u(x)\in m-cv(D)$, as $\delta\downarrow 0$ decreasing.

-- the limit of a uniformly convergent or decreasing sequence of $m-cv$ functions is $m-cv$;

-- the maximum of a finite number of $m-cv$ functions is $m-cv$ function; for an arbitrary locally uniformly bounded family $\{u_\theta\}\subset m-cv$, the regularization $u^*(x)$ of the supremum $u(x)=\left\{\sup\limits_\theta u_\theta(x)\right\}$ will also be $m-cv$ function. Since $m-cv\subset sh$, then the set $\{u(x)<u^*(x)\}$ is polar in $\mathbb{C}^n\approx\mathbb{R}^{2n}$. In particular, it has Lebesgue measure zero.

Similarly, for a locally uniformly bounded sequence $\{u_j\}\subset m-cv$, the regularization $u^*(x)$ of the limit $u(x)=\overline{\lim\limits_{j\to\infty}}u_j(x)$ will also be $m-cv$ function, and the set $\{u(x)<u^*(x)\}$ is polar;

-- if $u(x)\in m-cv(D)$, then for any hyperplane $\Pi\subset\mathbb{R}^n$ the restriction $u|_\Pi\in m-cv(D\cap\Pi)$.

In fact, considering, without loss of generality, $\Pi_x=\{x\in\mathbb{R}^n:\ x_n=0\}$ we write the restriction as $u|_\Pi=u('x,0)$, where as usual $'x=(x_1,...,x_{n-1})$. Consider a complex hyperplane $\Pi_z=\{z\in\mathbb{C}^n:\ z_n=0\}$ in the space $\mathbb{C}^n_z=\mathbb{R}^x_n\times i\mathbb{R}^y_n$. Raising the function $u(x)\in m-cv(D)$ in $D\times i\mathbb{R}^n_y$ we obtain $u^c(z)\in sh_m(D\times i\mathbb{R}^n_y)$. According to Property 8) [AS] the restriction $u^c(z)|_{\Pi_z}=u('z,0)$ is sh_m-function in $(D\times i\mathbb{R}^n_y)\cap\Pi_z$, $u^c(z,0)\in sh_m(D\times i\mathbb{R}^n_y)\cap\Pi_z$. Since $u^c('z,0)=u('x,0)$, then $u('x,0)$ is $m-$convex function in $D\cap\Pi_x$. ▹

Corollary. *If* $u(x)\in m-cv(D)$, *then for any plane* $\Pi\subset\mathbb{R}^n$, $\dim\Pi=m$, *the restriction* $u|_\Pi\in sh(D\cap\Pi)$.

4. In the class of bounded sh_m-functions, operators $(dd^cu)^k\wedge\beta^{n-k}\geq 0$, $k=1,2,...,n-m+1$ are defined as Borel measures in a domain D (see [B1], [AS]). Using the connection between $m-cv$ functions and sh_m-functions, in this section we give definitions of Hessians $H^k(u)$, $k=1,...,n-m+1$ for $m-$convex functions, like Borel measures.

Let $u(x)-$ be a locally bounded $m-cv$ function in the domain $D\subset\mathbb{R}^n$. Then, according to Theorem 1 $u^c(z)=u^c(x+iy)=u(x)$, which does not depend on variables $y\in\mathbb{R}^n_y$, is also a locally bounded sh_mfunction in the domain $D\times i\mathbb{R}^n_y\subset\mathbb{C}^n$. Consequently, the currents $(dd^cu^c)^k\wedge\beta^{n-k}$, $k=1,2,...,n-m+1$, are defined as Borel measures in $D\times i\mathbb{R}^n_y\subset\mathbb{C}^n$. If $u^c_j(z)=u_j\circ K(w-z)$ is the standard approximation, then $u^c_j(z)$ infinitely smooth and $u^c_j(z)\downarrow u^c(z)$. Moreover, there is weak convergence of currents,

$$\left(dd^c u_j^c\right)^k \wedge \beta^{n-k} \mapsto \left(dd^c u^c\right)^k \wedge \beta^{n-k},\ k = 1,2,...,n-m+1. \qquad (6)$$

Since $\left(dd^c u_j^c\right)^k \wedge \beta^{n-k} = k!(n-k)! H^k\left(u_j^c\right)\beta^n$, then (6) entails the convergence of Hessins

$$H^k\left(u_j^c\right) \mapsto H^k\left(u^c\right),\ k = 1,2,...,n-m+1. \qquad (7)$$

(7) defines for $u^c(z) \in sh_m\left(D \times i\mathbb{R}_y^n\right) \cap L_{loc}^\infty\left(D \times i\mathbb{R}_y^n\right)$ Hessians $H^k\left(u^c\right),\ k = 1,2,...,n-m+1,$ as Borel measures, $H^k\left(u^c\right) = \mu^k$.

Since $u^c \in sh_m\left(D \times \mathbb{R}_y^n\right)$ does not depend on $y \in \mathbb{R}_y^n$, then for any Borel sets $E_x \subset D,\ E_y \subset \mathbb{R}_y^n$ the measures $\frac{4^k}{mesE_y}\mu_k\left(E_x \times E_y\right)$ do not depend on the set, $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{4^k}{mesE_y}\mu_k\left(E_x \times E_y\right) = \nu_k\left(E_x\right)$. Borel measures

$$\nu_k:\ \nu_k\left(E_x\right) = \frac{4^k}{mesE_y}\mu_k\left(E_x \times E_y\right),\ k = 1,2,...,n-m+1, \qquad (8)$$

is natural, to call them Hessians $H^k(u),\ k = 1,2,...,n-m+1,$ for locally bounded, $m-$convex function $u(x) \in m-cv(D)$ in the domain $D \subset \mathbb{R}_x^n$, since, $H^k(u) = 4^k H^k\left(u^c\right)$ for a doubly smooth function $u(x) \in m-cv(D)$. Note that if a sequence $\left\{u_j(x)\right\} \subset m-cv(D)$ of locally uniformly bounded functions converges to $u(x)$, then there is weak convergence of measures $H^k\left(u_j\right) \mapsto H^k(u),\ k = 1,2,...,n-k+1,$ which easily follows from a similar fact for the class $sh_m\left(D \times i\mathbb{R}_y^n\right)$.

5. In conclusion of the report, I want to demonstrate two important and fundamental theorems that are proven using similar theorems in the class of sh_m-functions.

a) (comparison principle). **Theorem 2.** *Let* $u(x),v(x)\in m-cv(D)\bigcap L^{\infty}_{loc}(D)$ *and the set* $F=\{u(x)<v(x)\}\subset\subset D$. *Then*

$$\int_F H^{n-m+1}(u)\ge\int_F H^{n-m+1}(v). \tag{9}$$

Inequality (9) means that if for a domain $G\subset\subset D$ from $u|_{\partial G}=v|_{\partial G}$ and $u(x)<v(x)\ \forall x\in G$, then the total mass $\int_G H^{n-m+1}(v)\le\int_G H^{n-m+1}(u)$.

b) (maximal functions). **Definition 4.** A function $u(x)\in m-cv(D)$ is called *maximal* in the domain $D\subset\mathbb{R}^n$ if for this function the maximum principle holds in the class of $m-cv(D)$, i.e. if $v\in m-cv(D):\ \underline{\lim}_{x\to\partial D}(u(x)-v(x))$, then $u(x)\ge v(x),\ \forall x\in D$.

Note that the following convenient maximality criterion is often used: a function $u(x)\in m-cv(D)$ is maximal in the domain $D\subset\mathbb{R}^n$ if and only if for any domain $G\subset\subset D$ the inequality $u(x)\ge v(x), \forall x\in G$ holds for all functions $v\in m-cv(D):\ u|_{\partial G}\ge v|_{\partial G}$.

Maximal functions are closely related to the Dirichlet problem.

Theorem 3. *Let* $D=\{\rho(x)<0\}$ *strictly* $m-cv$ *convex domain in* $\mathbb{R}^n$ *and* $\varphi(\xi)$ *a continuous function defined on the boundary* ∂D. *Let's put*

$$\mathcal{U}(\varphi,D)=\{u\in m-cv(D)\bigcap C(\bar{D}):\ u|_{\partial D}\le\varphi\}$$

and

$$\omega(x)=\sup\{u(x):u\in\mathcal{U}(\varphi,D)\}. \tag{10}$$

Then, $\omega(x)\in m-cv(D)\bigcap C(\bar{D}),\ \omega|_{\partial D}=\varphi$ and in addition, $\omega(x)$ is the maximal $m-cv$ function in D.

We remember, a domain $D=\{\rho(x)<0\}$ is strictly $m-cv$ convex if the function $\rho(x)$ is strictly $m-cv$ in a neighborhood $D^+\supset\bar{D},\ \rho(x)\in m-cv(D^+),\ \rho(x)-\delta|x|^2\in m-cv(D^+)$ for some $\delta>0$.

It is natural to call the function $\omega(x)$ as a solution to the Dirichlet problem: $\omega(x)$ maximal and $\omega|_{\partial D}=\varphi$. Moreover, it is easy to see that a function $u\in m-cv(D)\bigcap C(D)$ is maximal if and only if the function $u^c(z)\in sh_m(D\times\mathbb{R}^n_y)\bigcap C(D\times\mathbb{R}^n_y)$ is a maximal sh_m function. It follows that $(dd^cu^c)^{n-m+1}\wedge\beta^{m-1}=0$ or $H^{n-m+1}(u^c)=0$. This is equivalent to $H^{n-m+1}(u(x))=0$.

[Proof of Theorem 3. Note that if in (8) instead of a class $m-cv(D)$ we consider a wider class of subharmonic functions $n-cv(D)=sh(D)\supset m-cv(D)$,

then we would obtain a solution to the classical Dirichlet problem: $v(x)=\sup\{u\in sh(D)\cap C(\bar{D}): u|_{\partial D}\le\varphi\}$. In this case $\Delta v\equiv 0,\ v|_{\partial D}\equiv\varphi$. It is clear that $\omega(x)\le v(x)$ and

$$\overline{\lim_{x\to\xi}}\,\omega(x)\le\varphi(\xi),\ \ \forall\xi\in\partial D. \qquad (9)$$

On the other hand, any fixed boundary point $\xi^0\in\partial D$ of a strictly $m-$convex domain $D=\{\rho(x)<0\}$, $\rho(x)-$strictly $m-cv$ function in some neighborhood $D^+\supset\bar{D}$, is a peak point: there exists $v\in m-cv(D)\cap C(\bar{D}):\ v(\xi^0)=0,\ v|_{\bar{D}\setminus\{\xi^0\}}<0$.

In fact, since $\rho(x)$ strictly $m-cv$ function in a certain neighborhood $D^+\supset\bar{D}$, then for a sufficiently small positive number $\delta>0$ the difference $\rho(x)-\delta\left\|x-\xi^0\right\|^2$ is $m-$convex in D^+. Considering instead $\rho(x)$ function

$$v(x)=\rho(x)-\delta\left\|x-\xi^0\right\|^2\in m-cv(D)\cap C(\bar{D}): v(\xi^0)=0,\ \ v|_{\bar{D}\setminus\{\xi^0\}}<0$$

we'll make sure that the point $\xi^0\in\partial D$ is peak point.

Hence, for any fixed number $\varepsilon>0$ there is a large number $M>0$ that $M\cdot v(x)+\varphi(\xi^0)-\varepsilon\in\mathcal{U}(\varphi,D)$. Therefore, $M\cdot v(x)+\varphi(\xi^0)-\varepsilon\le\omega(x)$ and $\underline{\lim}_{x\to\xi^0}\,\omega(x)\ge\varphi(\xi^0)-\varepsilon$. Since the number $\varepsilon>0$ and point $\xi^0\in\partial D$ are arbitrary, then $\underline{\lim}_{x\to\xi}\,\omega(x)\le\varphi(\xi),\ \forall\xi\in\partial D$. Combining this with (9) we get $\lim_{x\to\xi}\omega(x)=\varphi(\xi),\ \forall\xi\in\partial D$.

For regularization ω^* which is $m-cv$ function in the domain D condition of continuity on the boundary is also satisfied: $\lim_{x\to\xi}\omega^*(x)=\varphi(\xi),\ \forall\xi\in\partial D$. From $\omega^*(x)\in m-cv(D),\ \lim_{x\to\partial D}\omega^*=\varphi$ follows that $\omega^*(x)\le\omega(x)$, i.e. $\omega^*(x)\equiv\omega(x)$ and $\omega(x)$ is $m-cv$ function. Let us show that it is maximal.

Assume the contrary, let there be a domain $G\subset\subset D$ and a function $\phi(x)\in m-cv(D):\phi|_{\partial G}\le\omega|_{\partial G}$, but $\phi(x^0)>\omega(x^0)$ at some point x^0.

Function

$$w(x)=\begin{cases}\max\{\omega(x),\phi(x)\} & if\ \ x\in\bar{G}\\ \omega\quad if\ \ x\in D\setminus G\end{cases}$$

is $m-$convex, $w(x)\in m-cv(D),\ w|_{\partial D}=\omega|_{\partial D}=\varphi$. Therefore, $w(x)\le\omega(x)$ and $\phi(x^0)\le\omega(x^0)$. This is contradiction.

It remains to prove that the function ω will be continuous in the closure. Let's build an approximation

$$\omega_\delta(x)=\omega\circ K_\delta(x-y)\in m-cv(D_\delta)\cap C^\infty(D_\delta),\ D_\delta=\{x\in D:\rho(x)<\delta\},$$

$\omega_\delta(x)\downarrow\omega(x),$ as $\delta\downarrow 0$. For small enough $\delta>0$ each interior normal $n_\xi,\ \xi\in\partial D$ intersects ∂D_δ at a single point $\eta(\xi)\in\partial D_\delta$, so that a homeomorphism n_δ is defined $n_\delta:\partial D\to\partial D_\delta$. Let us put $\varphi_\delta(\eta)=\varphi(n_\delta(\xi))$, $\eta\in\partial D_\delta,\ \xi\in D$. Since $\lim\limits_{x\to\xi}\omega(x)=\varphi(\xi),\ \forall\xi\in\partial D$, then for any fixed $\varepsilon>0$ there is a $\delta_0>0$ such that $\left|\omega(x)-\varphi_{\delta_0}(x)\right|<\varepsilon,\ \forall x\in\partial D_{\delta_0}$. For a fixed $\delta_0>0$ the domain $D_{\delta_0}\subset\subset D$ and the approximation $\omega_\delta(x)\downarrow\omega(x),$ for $\delta\downarrow 0$ covers the domain D_{δ_0}.

Now applying Hartogs' lemma to a compact set ∂D_{δ_0} and a function $\varphi_{\delta_0}(x)\in C(\partial D_{\delta_0})$ we find $0<\delta'<\delta_0$ such that

$$\omega_\delta(x)<\omega_{\delta_0}(x)+3\varepsilon,\ \forall x\in\partial D_{\delta_0},\ \delta<\delta'. \tag{10}$$

Since the solution to the Dirichlet problem $\omega(x)-$is maximal in D, from $\omega_\delta(x)<\varphi_{\delta_0}(x)+3\varepsilon, \forall x\in\partial D_{\delta_0},\delta<\delta'$ follows that $\omega_\delta(x)<\omega(x)+4\varepsilon,\ \forall x\in D_{\delta_0}$, $\delta<\delta'$ because $\omega(x)>\varphi_{\delta_0}(x)-3\varepsilon,\ \forall x\in\partial D_{\delta_0}$. From here, $\omega(x)<\omega_\delta(x)<\omega(x)+4\varepsilon,\ \forall x\in\partial D_{\delta_0},\ \delta<\delta',$ i.e. $\left|\omega_\delta(x)-\omega(x)\right|<4\varepsilon,$ $\forall x\in D_{\delta_0},\ \delta<\delta'(\delta_0).$ Since $\varepsilon>0$ arbitrary, then the convergence $\omega_\delta(x)\downarrow\omega(x)$ will be uniform inside D and $\omega(x)\in C(D)$, because $\omega_\delta(x)\in C^\infty(D_\delta)$. *The theorem is proven.]*

Theorem 4. *A continuous $m-cv$ function $u(x)\in m-cv(D)\bigcap C(D)$ is maximal if and only if the Borel measure is $H_u^{n-m+1}=0$.*

Proof. We proved above the equality $H_u^{n-m+1}=0$ for the maximal function $u(x)\in m-cv(D)\bigcap C(D)$. Let now $H_u^{n-m+1}=0$ and we will show that u maximal. Assume that u is not the maximal. Then for some domain $G\subset\subset D$ there is a function $v\in m-cv(D):\ u|_{\partial G}\geq v|_{\partial G}$, but $v(x^0)-u(x^0)=\varepsilon>0$ for some point $x^0\in G$.

Approximating v by infinitely smooth $m-cv$ functions $v_j\downarrow v$, and then using Hartog's lemma, we find $j_0\in\mathbb{N}$ such that $v_{j_0}\big|_{\partial G}<u|_{\partial G}+\dfrac{\varepsilon}{2}$. Let us compare the function $u(x)$ with the function $v_{j_0}(x)+\delta\|x\|^2$, where $\delta=\dfrac{\varepsilon}{3\cdot\max\left\{\|x\|^2:x\in\overline{G}\right\}}$.

For such $\delta>0$ a set $F=\left\{u(x)+\dfrac{\varepsilon}{2}<v_{j_0}(x)+\delta\|x\|^2\right\}$ is not empty and lies compactly in G. Then according to the comparison principle (Theorem 2)

$$\delta^n\int_F\left(dd^c\|x\|^2\right)^n\leq\int_F\left(dd^c v+\delta dd^c\|x\|^2\right)^n\leq\int_F\left(dd^c u\right)^n=0,$$

which contradicts to $\int_F \left(dd^c \|x\|^2\right)^n > 0$. *The theorem is proven.*

Notation. *In Theorem 4 we required the continuity of function* $u(x) \in m-cv(D) \cap C(D)$ *for simplicity of the proof. In fact, it is also true for functions* $u(x) \in m-cv(D) \cap L^\infty_{loc}(D)$*: a function* $u(x) \in m-cv(D) \cap L^\infty_{loc}(D)$ *is maximal if and only if the Hessian* $H^{n-m+1}(u) = 0$ *in* $D \subset \mathbb{R}^n$.

c) *let* $u(x) \in m-cv(B) \cap L^\infty(B)$, *where* $B = \{x \in \mathbb{R}^n : \ |x| < 1\}$ *be the unit ball.*

Then, we have the following estimate for the Hessian $H_u^k(x)$ *in terms of the Hessian of a lesser degree*

$$\int_{-1}^{r} dt \int_{|y|^2 \le 1} dV(y) \int_{|x|^2 \le 1+t-|y|^2} H_u^k(x) dV(x) \le \le \frac{n-k+1}{k}(M-m) \int_{|y|^2 \le 1} dV(y) \int_{|x|^2 \le 1+r-|y|^2} H_u^{k-1}(x) dV(x), \tag{11}$$

where $r < 0,\ 1 \le k \le n-m+1$ *and* $M = \sup_B u(z),\ m = \inf_\Omega u(z)$.

Applying (11) $k-$times we have

Corollary. *In the class of locally uniformly bounded functions* $L = \{u(x) \in m-cv(D)\}$, *the family of integrals* $\int_K H_u^k(x) dV(x),\ u \in L,$ *is uniformly bounded for any compact set* $K \subset D,\ 1 \le k \le n-k+1,$

$$\int_K H_u^k(x) dV(x) \le C(K),\ u \in L.$$

Thank you for your attention

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