Independence polynomial on arbitrary recursive graphs

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Auf dem heiligen berg
Wuppertal

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Summary and conclusions

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These dynamical systems have common features, with consequences for partition functions.

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Corollary: For G_0 *maximally independent* the zeros of the independence polynomials are uniformly bounded.

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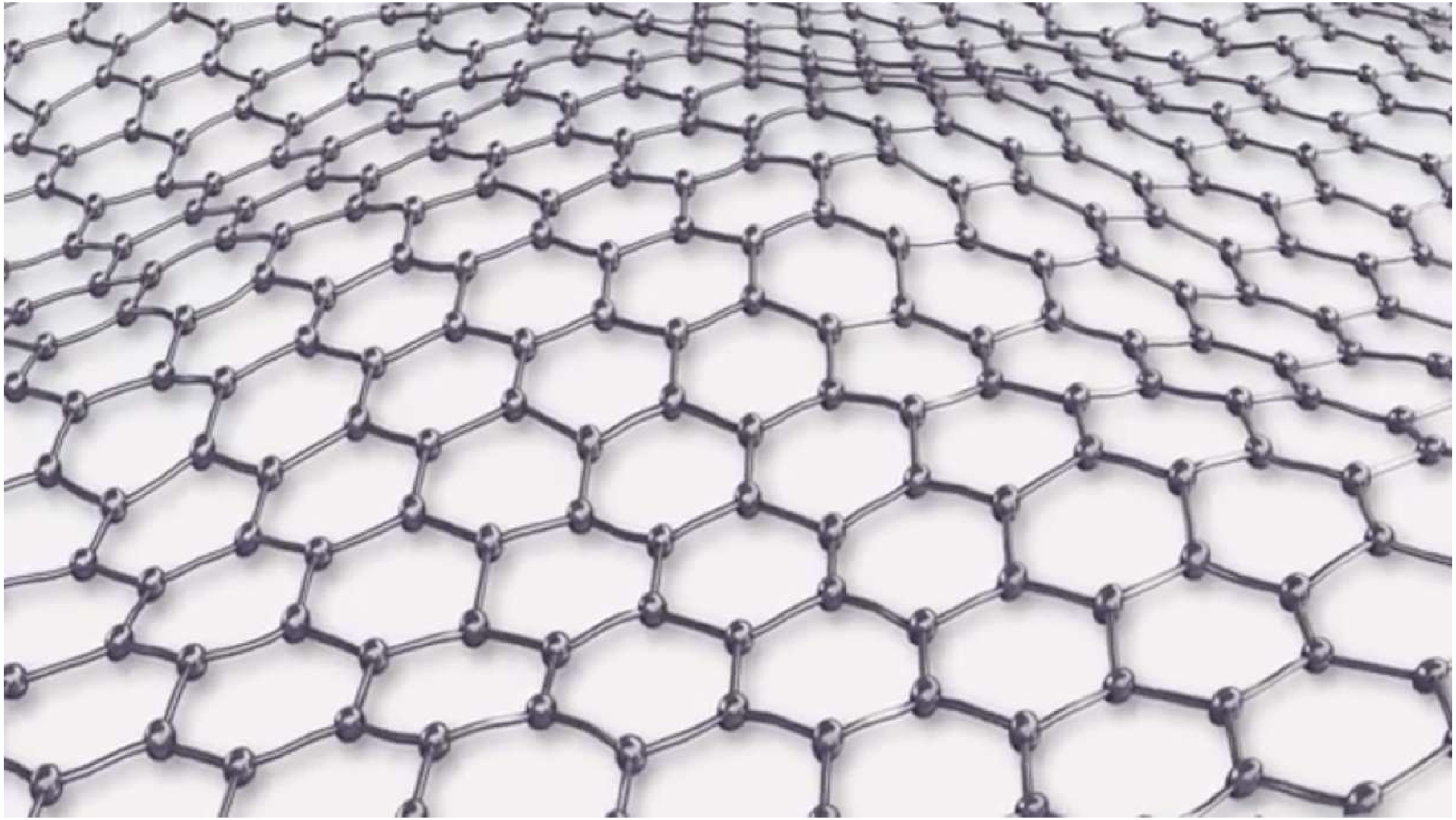

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Key idea: The almost infinite system is a limit of larger and larger finite systems.

Two spin models on graphs

Assume interaction energies are constant. Obtain a *graph* G and states $\sigma : G \to \{spins\}$.
Further assume there are only two spins, say $\{0, 1\}$.

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Hard-core model

Let

$$Z_G(\lambda) = \sum_{\sigma \text{ ind.}} \prod_{v \in V(G)} \lambda^{\sigma(v)},$$

summing over *independent* σ: $\sigma(v) \cdot \sigma(w) = 0$ for every $(v, w) \in E(G)$.

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Z_G is called the *independence polynomial*.

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Related are the *Tutte polynomial* and the *Chromatic polynomial*, which consider q as a parameter.

Modeling infinite graphs as limits of a sequence (G_n)

To each graph G_n we associate a normalized free energy:

$$\rho_n(\lambda) = \frac{\log |Z_G(\lambda)|}{|V(G_n)|}$$

The free energy of the limiting system is the limit of ρ_n as $n \to \infty$.

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Yang-Lee (1952)

If the zeros of the polynomials $Z_{G_n}(\lambda)$ avoid a **complex neighborhood** of the parameter λ_0, then the limiting free energy is real analytic at λ_0.

Partition functions on regular lattices

Folklore Conjecture

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How can it be that such a simple question is still open?

1. Regular lattices are not trivial.
2. Computation of G_n is “hard”.

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Consider all graphs with vertex degrees bounded by $\Delta \geq 2$. Let $0 \in U$ be the maximal zero-free domain.

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A precise description of U is still lacking.

Relevance of zero-sets of partition functions

To summarize

1. Zeros are related to phase transitions.
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A recent result:

de Boer-Buys-P.-Regts, 2024

Consider an increasing sequence of d-dimensional torus-graphs. If the tori are balanced, the zeros are bounded. If the tori are highly unbalanced, the zeros are unbounded.

Example of recursive graphs, I

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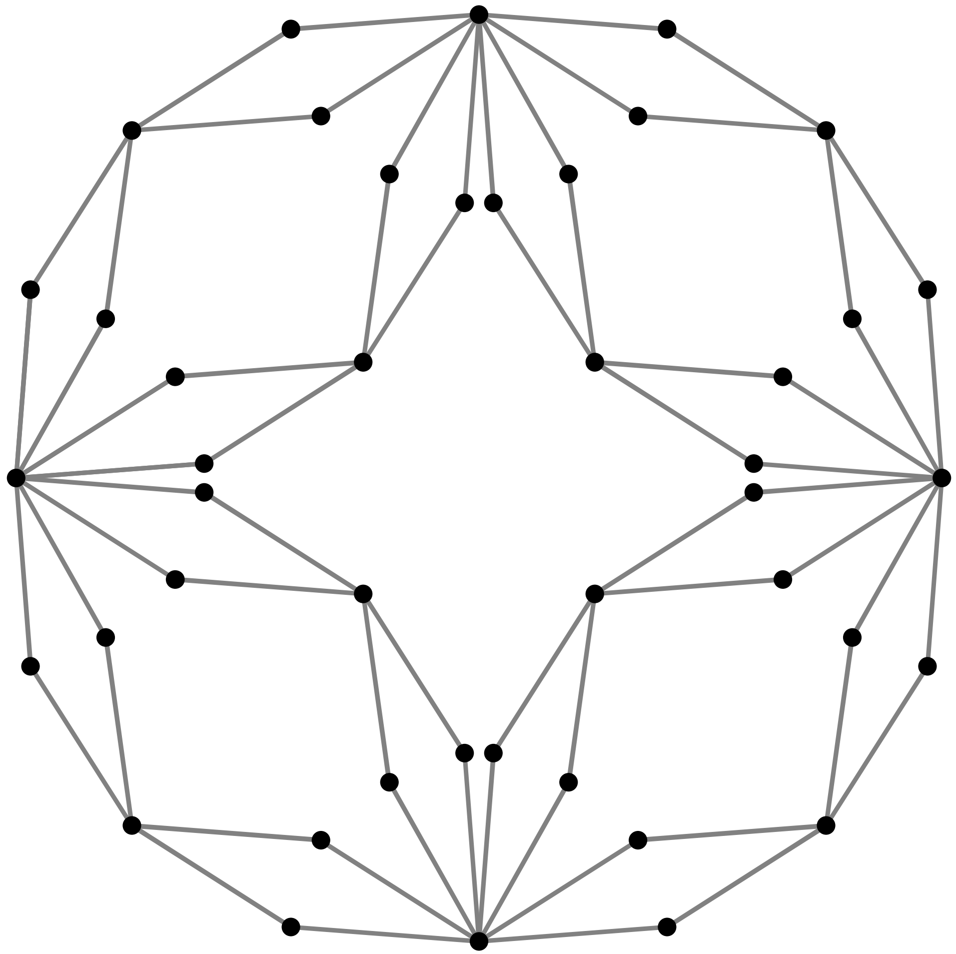
Bleher-Lyubich-Roeder (2010), Chio-Roeder (2021)

Consider the Ising model on *diamond hierarchical lattices*. Then there is a unique phase transition.



Example of recursive graphs, II

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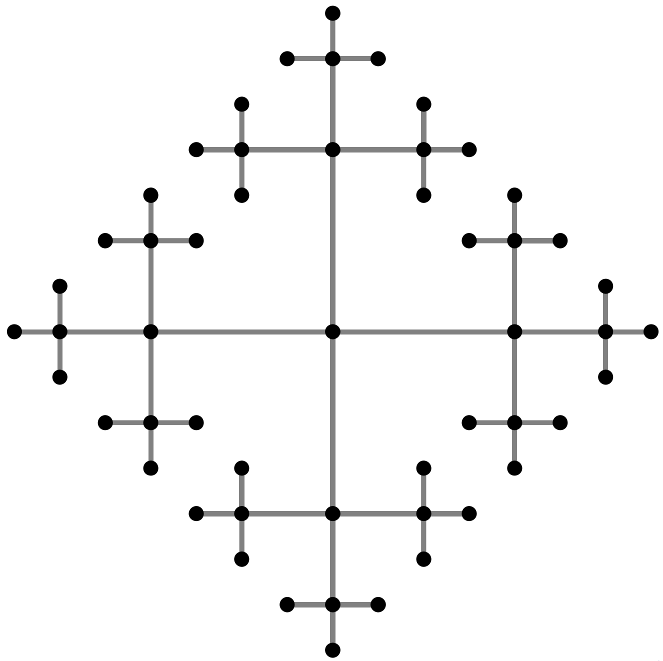
Rivera-Letelier& Sombra (talk at Fields Institute, 2019)

Consider the Hard-Core model on *d-ary trees*. Then zeros accumulate at a unique parameter in $\mathbb{R}_+$:

$$\lambda(d) = \frac{d^d}{(d+1)^{d-1}},$$

the unique phase transition **of infinite order**.



Example of recursive graphs, III

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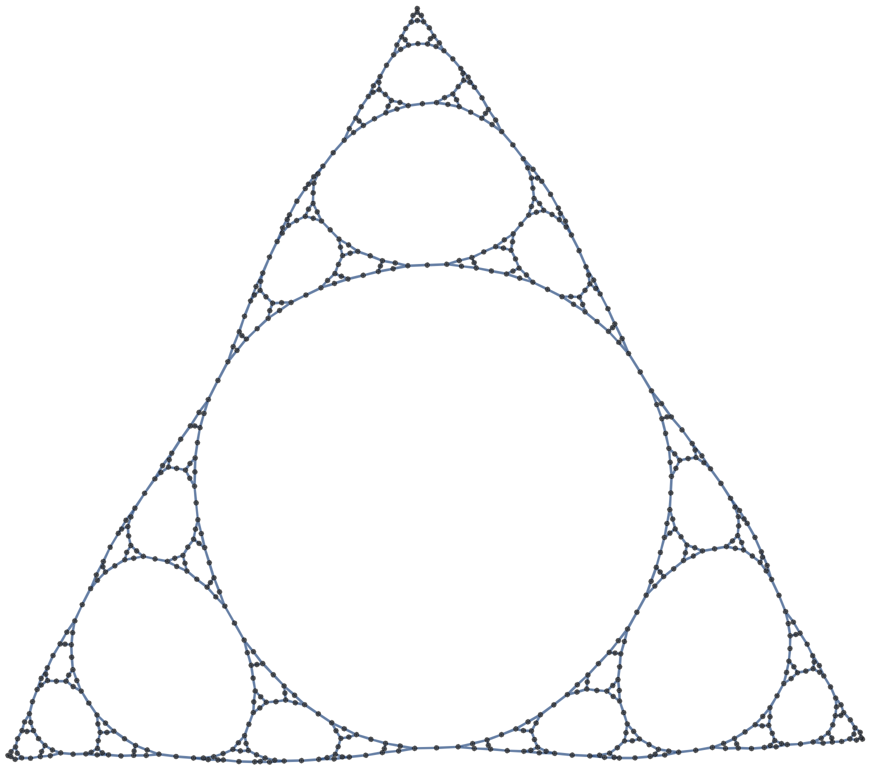
Nguyen-Bac Dang, Rostislav Grigorchuk, Mikhail Lyubich, 2021

Spectrum of the Laplacian on Schreier graphs of some self-similar groups.



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Partition functions and recursive graphs

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In all of these examples, the recursion induces a rational dynamical system, which can be studied to describe the zeros.

The purpose of this project is to present a **general framework**, to study the induced dynamical systems, and to draw conclusions regarding the partition functions.

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We call (H, Σ, Φ) the *gluing data*.

Constructing the sequence of graphs

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Let G_{n+1} be the obtained graph.

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Step 4. Mark k vertices of G_{n+1} using the function $\Phi : \{1, \ldots, k\} \to E(H)$. If $e = \Phi(j)$ has multiple vertices, label the marked vertex of Σ_e.

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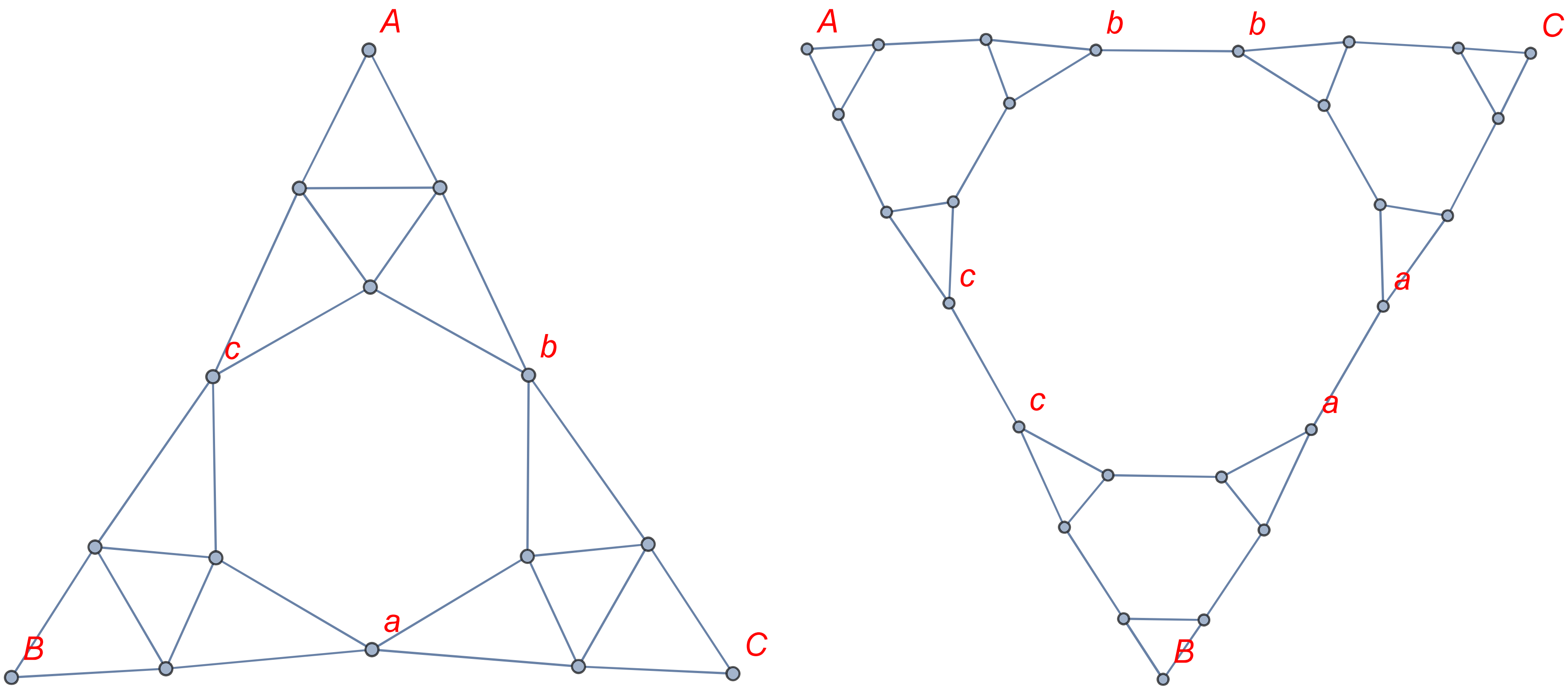
Step 4. Define $\Phi(j) = \{v_j(j)\}$.



The Sierpinsky triangle G_2 and the towers of Hanoi G_2, where the connecting graphs are *edges*.

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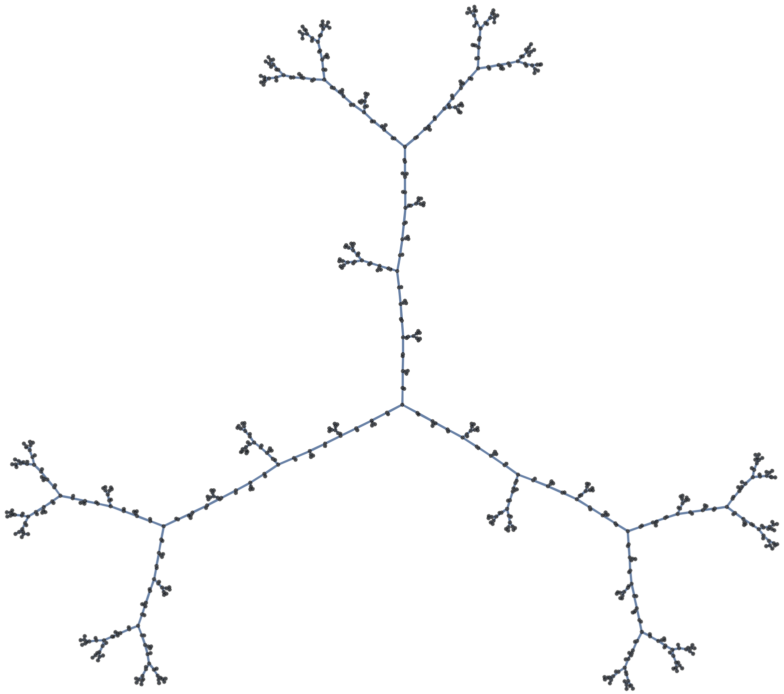
Step 4. Let $\Phi(1) = \{v_1(1)\}$ and $\Phi(2) = \{v_3(1)\}$.



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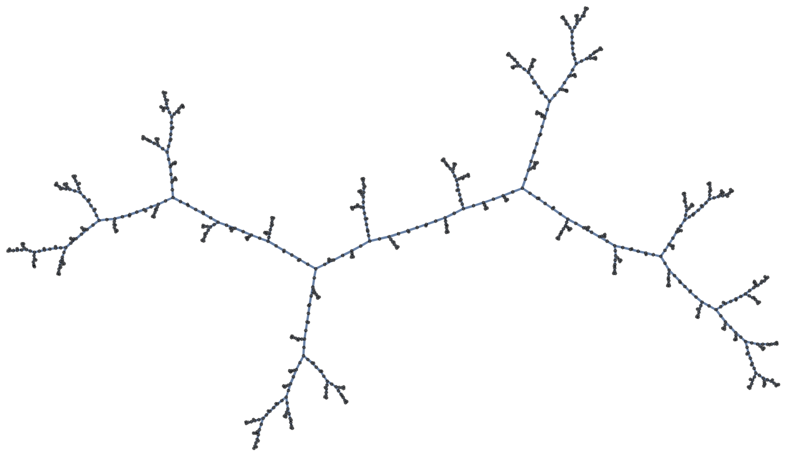
Step 4. Let $\Phi(1) = \{v_1(3)\}$, $\Phi(2) = \{v_1(1)\}$ and $\Phi(3) = \{v_2(1)\}$.



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Write

$$Z_{G_n}(\lambda) = \sum_{(x_1,\ldots,x_k)\in\{0,1\}^k} (x_1,\ldots,x_k)_n,$$

where

$$(x_1,\ldots,x_k)_n = Z_{G_n}(\lambda, x_1,\ldots,x_k)$$

sums only $\sigma : V(G_n) \to \{0,1\}$ with $\sigma(j) = x_j$.

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Each $(x_1, \dots, x_k)_{n+1}$ can be expressed in the variables $(y_1, \dots, y_k)_n$ as a homogeneous polynomial of degree m.

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Formula for $(x_1,\dots,x_k)_{n+1}$:

$$\sum_{x\sim y\in\{0,1\}^{km}} \prod_{i=1}^{m} (y_1(i),\dots,y_k(i)) \cdot \prod_{e\in E(H)} \frac{Z_{\Sigma_e}(\lambda, y|_e, x|_e)}{\lambda^{|y|_e|}}$$

An invariant manifold

Observation

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Hence the equations

$$\frac{(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_k)}{(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_k)} = \frac{(0, \ldots, 0, 1, 0, \ldots, 0)}{(0, \ldots, 0, 0, 0, \ldots, 0)}$$

define a k-dimensional manifold in $\mathbb{C}^{2^k}$ and in $\mathbb{P}^{2^k-1}$.

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As a consequence, the invariant 3-manifold is mapped onto a periodic 2-manifold, which is a graph over the variables

$$[1,0,0]_n = \frac{(1,0,0)_n}{(0,0,0)_n} \quad \text{and} \quad [0,0,1]_n = \frac{(0,0,1)_n}{(0,0,0)_n}$$

For the second iterate this 2-manifold in $\mathbb{P}^2$ consists of fixed points.

Understanding the dynamics near the fixed manifold

Assume that none of the periodic labels are *critical*.

Theorem

The periodic manifold is normally super-attracting.

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Proof by Mathematica.

```
In[21]:= Eigenvalues[jacobiansurface]

Out[21]= {0, 0, 0, 0, 0, 1, 1}
```

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Definition

A labeled graph G_n is *maximally independent* if for every assignment $x = (x_1, \ldots, x_k)$ the maximal independent $I(x) \subset V(G_0)$ is unique, and moreover

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For the Antenna recursion the edge G_0 is not maximally independent, but G_2 is.

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Since the fixed manifold is normally super-attracting, the orbit of z_0 will be contracted towards the manifold.

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For any *maximally independent* G_0 and *non-degenerate* recursion (H, Σ, Φ) the zeros of the independence polynomials of (G_n) are uniformly bounded.

Idea of the proof For large λ the initial value in $\mathbb{CP}^{2^k-1}$ will lie close to the fixed manifold.

Since the fixed manifold is normally super-attracting, the orbit of z_0 will be contracted towards the manifold.

When summing the coordinates, the terms $(1, \ldots, 1)_n$ will dominate the others, hence no zeros. $\square$

Conclusion and future work

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This is just a start:

1. When are zeros bounded away from $\mathbb{R}_+$?
2. When do zeros equidistribute?
3. Does the behavior depend on (G_0, Σ), or only on (H, Φ)?
4. What about other partition functions?
5. ...

Thank you.