Invariant curves in strictly pseudoconvex boundaries

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Strictly pseudoconvex boundaries

Assume that $\Omega \subset \mathbb{C}^n$ is a smoothly bounded strictly pseudoconvex domain. Then we first recall the following notions:

1. Extremal and stationary discs;
2. Chern-Moser chains;
3. Segre varieties (if the boundary is, in addition, real analytic).

Extremal and stationary discs

For $p \in \Omega$ and $v \in T_p\Omega$, an *extremal* disc at p in the direction of v is a solution $f\colon \mathbb{D} \to \Omega$ of the extremal problem

$$\sup\{|g'(0)|: g\colon \mathbb{D} \to \Omega,\ g(0) = 0,\ g'(0) = \lambda v \text{ for some } \lambda > 0\}.$$

- Lempert (1981): extremal discs extend smoothly to the boundary if Ω is convex.
- Every extremal disc is *stationary*, meaning that there exists a lift $(f, \tilde{f})$, holomorphic except for a pole of order 1 at the origin, which is attached to the conormal bundle $T^0 M$.
- Huang (1994) Localization for strictly pseudoconvex boundaries: For every $p_0 \in \partial\Omega$, there exist a $\varepsilon > 0$ such that extremal discs at points p on the inward normal at p_0 in directions v such that $|v - \pi_{T^c_{p_0}} v| < \varepsilon |v|$ satisfy the same conclusion.

Invariance properties

1. Extremal discs are invariant: If $h \colon \Omega \to \Omega'$ is a holomorphic map, and f is extremal, then $h \circ f$ is extremal.
2. Stationary discs are invariant: If $h \colon \Omega \to \Omega'$ is a holomorphic map and f is stationary, then $h \circ f$ is stationary.
3. The latter statement actually nicely localizes as well.

Invariant curves

In particular, the boundary traces of stationary discs give CR invariant curves in $\partial\Omega$.

An example

For the sphere $|z_1|^2 + |z_2|^2 = 1$, extremal discs are intersections with complex planes. Computing with "Heisenberg coordinates", $\varrho(z, w) = \operatorname{Im} w - |z|^2$:

$$\partial\varrho = (-\bar{z}, \frac{1}{2i}).$$

If $(f, \tilde{f})$ is the lift of a stationary disc, then there exists an $h > 0$ defined on $\partial\mathbb{D}$ such that $\tilde{f} = h\zeta\partial\varrho$ extends holomorphically to $\mathbb{D}$, and

$$\int \zeta^n h(\zeta)\partial\varrho(f(\zeta)) = 0, \quad n > 1.$$

This implies that $h(\zeta)$ is constant and thus $\zeta\bar{f}(\bar{\zeta}) = \zeta\bar{f}\left(\frac{1}{\zeta}\right)$ only has positive Fourier coefficients. Thus $f_1(\zeta) = a + b\zeta$, and $|a + b\zeta|^2 = |a|^2 + a\bar{b}\bar{\zeta} + \bar{a}b\zeta + |b|^2 z\bar{z} = |a|^2 + |b|^2 + a\bar{b}\zeta + \bar{a}b\bar{\zeta} =$ $\operatorname{Im}\left(i(|a|^2 + |b|^2) + a - ib\zeta\right)$

Invariant geometry I: Cartan

Given a strictly pseudoconvex hypersurface in $\mathbb{C}^2$, it comes with the distribution $T^c M = (T^0 M)^\perp = \langle \theta \rangle^\perp$.

$$h(X, Y) = \theta([X, \bar{Y}]) = -d\theta(X, \bar{Y}),$$

Positive definite (Levi form associated to θ.)

$$E = \langle u\theta \colon u > 0 \rangle, \quad \omega = u\theta.$$

On E :

$$d\omega = i g_{1\bar{1}} \omega^1 \wedge \bar{\omega}^1 + \omega \wedge \varphi,$$

with $\omega, \omega^1, \bar{\omega}^1, \varphi$ spanning $T^* E$. Freedom for such frames:

$$\begin{pmatrix} \tilde{\omega} \\ \tilde{\omega}^1 \\ \bar{\tilde{\omega}}^1 \\ \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda & \mu & 0 & 0 \\ \bar{\lambda} & 0 & \bar{\mu} & 0 \\ s & i\mu\bar{\lambda} & -i\bar{\mu}\lambda & 1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega^1 \\ \bar{\omega}^1 \\ \varphi \end{pmatrix}, \quad |\mu|^2 = 1.$$

Yields bundle Y over E.

Theorem (Cartan 1932)

There exists a unique frame $\omega, \omega^1, \bar\omega^1, \varphi, \alpha, \beta, \psi$ *of* $\mathbb{C}T^*Y$ *and invariantly defined functions* Q, R *such that*

$$d\omega = i\omega^1 \wedge \bar\omega^1 + \omega \wedge \varphi, \quad d\omega^1 = \omega^1 \wedge \alpha + \omega \wedge \beta, \quad \varphi = \alpha + \bar\alpha,$$

$$d\varphi = i\omega^1 \wedge \bar\beta - i\bar\omega^1 \wedge \beta + \omega \wedge \psi, \quad d\alpha = i\bar\omega^1 \wedge \beta + 2i\omega^1 \wedge \bar\beta - \frac{\psi}{2} \wedge \omega,$$

$$d\beta = \bar\alpha \wedge \beta - \frac{\psi}{2} \wedge \omega^1 + Q\bar\omega^1 \wedge \omega, \quad d\psi = \varphi \wedge \psi + 2i\beta \wedge \bar\beta + (R\omega^1 + \bar R\bar\omega^1) \wedge \omega.$$

Definition

A curve γ is called a *chain* if it solves the system of ODEs $\omega^1 = \beta = 0$.

Note the equations for the real forms ω, φ, and ψ along a chain:

$$d\omega = \omega \wedge \varphi, \quad d\varphi = \omega \wedge \psi, \quad d\psi = \varphi \wedge \psi$$

giving a canonical parameter along the chain (up to a linear fractional map).

Invariant geometry II: Chern-Moser

$$\operatorname{Re} z_1 = |z_2|^2 + \varphi(z_2, \bar{z}_2, \operatorname{Im} z_1) = |z_2|^2 + \sum_{\alpha,\bar{\beta}} \varphi_{\alpha,\bar{\beta}}(\operatorname{Im} z_1) {z_2}^{\alpha} {\bar{z}_2}^{\bar{\beta}}.$$

Theorem (Chern-Moser normal coordinates)

Let (M, p) be a real-analytic hypersurface. Then there exist holomorphic coordinates (z, w) in which $p = 0$ and $\varphi_{\alpha,\bar{\beta}}(\operatorname{Im} z_1) = 0$ if $\min(\alpha, \bar{\beta}) \leq 1$ or $(\alpha, \beta) \in \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$. Any other choice $(\tilde{z}, \tilde{w}) = H_\Lambda(z, w)$ for some $\Lambda = (\lambda, a, t) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R}$, with H_Λ uniquely determined by the requirement that it agrees with the map

$$(z_1, z_2) \mapsto \left(\frac{|\lambda|^2 z_1}{1 + 2\bar{a} z_2 + (|a|^2 + it) z_1}, \frac{\lambda(z_2 + a z_1)}{1 + 2\bar{a} z_2 + (|a|^2 + it) z_1} \right)$$

up to order two.

Chains in Chern-Moser

Chains in Normal Coordinates

In Chern-Moser normal coordinates, the line $z_2 = 0$ in M is a chain. Conversely, every chain is realized in that way.

Remarks

- The geometric data determining a chain is essentially a direction transverse to T^cM and a vector from T^cM; roughly a and λ.
- Yields a curve γ with a frame of T^cM along γ.
- The parameter change roughly corresponds to the t.

Umbilicity and sphericity

A point p is umbilic if $\varphi_{4,\bar{2}}(0) = 0$ in one (and hence every) set of normal coordinates at p. M is locally spherical if it is umbilic at every point.

An example

Recalling that the automorphism group of the sphere given in the form

$$\operatorname{Re} z_2 = |z_1|^2$$

is generated by the maps

$$(z_1, z_2) \mapsto \left(\frac{|\lambda|^2 z_1}{1 + 2\bar{a} z_2 + (|a|^2 + it) z_1}, \frac{\lambda(z_2 + a z_1)}{1 + 2\bar{a} z_2 + (|a|^2 + it) z_1} \right)$$

together with the translations

$$(z_1, z_2) \mapsto (z_1 + a, z_2 + 2\bar{a} z_1 + |a|^2 + ir),$$

one can check that the intersection with any complex line can be brought into the form $z_1 = 0$, and thus the chains are exactly the intersections with complex lines.

Invariant geometry III: Fefferman

Considering $\rho = x_1 - (x_2^2 + y_2^2) - \varphi(x_1, x_2, y_2)$ as the defining equation of M, one can iteratively construct approximate solutions to the Monge-Ampere equation

$$J(\rho^{(k)}) = \det \begin{pmatrix} \rho^{(k)} & \rho^{(k)}_{\bar{z}_1} & \rho^{(k)}_{\bar{z}_2} \\ \rho^{(k)}_{z_1} & \rho^{(k)}_{z_1\bar{z}_1} & \rho^{(k)}_{z_1\bar{z}_2} \\ \rho^{(k)}_{z_2} & \rho^{(k)}_{z_2\bar{z}_1} & \rho^{(k)}_{z_2\bar{z}_2} \end{pmatrix} = 1 + O(\rho^{k+1})$$

$$\rho^{(1)} = \frac{\rho}{\sqrt[3]{J(\rho)}}, \quad \rho^{(2)} = \rho^{(1)} \left(\frac{5 - J(\rho^{(1)})}{4} \right).$$

$\rho^{(2)}$ can be used to construct a nice metric on a circle bundle $\mathbb{S} \times M$ over M, here with coordinates (x_0, x_1, x_2, y_2):

$$ds^2 = -\frac{i}{3} \left(\partial\rho^{(2)} - \bar{\partial}\rho^{(2)} \right) dx_0 + \sum_{j,k=1}^{2} \frac{\partial^2 \rho^{(2)}}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k.$$

The Hamiltonian

To compute the light rays of this metric, whose projections to M are again the chains,

$$\Phi = J(\rho), \quad A = \begin{pmatrix} 0 & i\rho_{\bar{z}_1} & i\rho_{\bar{z}_2} \\ -i\rho_{z_1} & 3\rho_{z_1\bar{z}_1} & 3\rho_{z_1\bar{z}_2} \\ -i\rho_{z_2} & 3\rho_{z_2\bar{z}_1} & 3\rho_{z_2\bar{z}_2} \end{pmatrix}, \quad P = (p_{x_0}, ip_{y_1}, p_{x_2} + ip_{y_2})$$

$$\overline{\partial}\Phi = \left(0, \Phi_{\bar{z}_1}, \Phi_{\bar{z}_2}\right), \quad \tilde{\Phi} = \left(3\Phi_{j\bar{k}} - \frac{5}{\Phi}\Phi_j\Phi_{\bar{k}}\right)_{j,k}$$

allows us to compactly write its Hamiltonian

$$H = PA^{-1}P^* - \frac{2p_{x_0}}{\Phi}\operatorname{Im}\left(\overline{\partial}\Phi \cdot A^{-1} \cdot P^*\right) - \frac{p_{x_0}^2}{2\Phi}\operatorname{Tr}\left(\tilde{A}^{-1}\right).$$

An example

The equations

$$H(x,p)=0, \quad x'=H_p(x,p), \quad p'=-H_x(x,p)$$

now give us a very concrete way to compute the chains. In the case of the sphere, one obtains

$$\begin{cases} 0 = 6p_{x_0}p_{y_1} - |z_2|^2 p_{y_1}^2 + 2y_2 p_{y_1} p_{x_2} - 2x_2 p_{y_1} p_{y_2} - p_{x_2}^2 - p_{y_2}^2 \\ x_0' = 6p_{y_1} \\ y_1' = 6p_{x_0} - 2|z_2|^2 p_{y_1} + 2y_2 p_{x_2} - 2x_2 p_{y_2} \\ x_2' = 2y_2 p_{y_1} - 2p_{x_2} \\ y_2' = -2x_2 p_{y_1} - 2p_{y_2} \\ p_{x_0}' = 0 \\ p_{y_1}' = 0 \\ p_{x_2}' = 2x_2 p_{y_1}^2 + 2p_{y_1} p_{y_2} \\ p_{y_2}' = 2y_2 p_{y_1}^2 - 2p_{y_1} p_{x_2} \end{cases}$$

Summarizing: Chains

In summary, the family of chains is another family of local CR invariant curves. They can be obtained in a number of ways:

- Intrinsically through the complete parallelism from Cartan;
- Extrinsically through Chern-Moser normal coordinates;
- Extrinsically through Fefferman's metric.

We remark that Fefferman used his approac to provide an example of "spiraling" chains.

Segre varieties

If M is real analytic, given by

$$\varrho(z_1, z_2, \bar{z}_1, \bar{z}_2) = 0,$$

then the Segre variety associated to the point p is the complex variety given by

$$S_p = \{(z_1, z_2) \colon \varrho(z_1, z_2, \bar{p}_1, \bar{p}_2) = 0\}.$$

They have the property that

$$q \in S_p \Leftrightarrow p \in S_q, \quad p \in S_p \Leftrightarrow p \in M, \quad H \colon M \to M \Rightarrow H(S_p) \subset S_{H(p)}.$$

For $p \notin M$ near M on the concave side, we have that S_p intersects M transversally in a closed curve (which is everywhere transverse to the complex tangent space).

Traces of Segre varieties; and an example

The traces of Segre varieties therefore yield a third family of local CR invariants (if M is real analytic of course).
If M is the sphere, given by $\operatorname{Re} z_1 = |z_2|^2$, the Segre variety associated to $p = (\bar{a}, \bar{b})$ is given by $z_1 + a = 2 z_2 b$. In other words, the Segre varieties are just complex lines ... again.

An observation and an old result

Therefore, if M is the sphere, the three invariant families:

1. Boundaries of stationary discs;
2. Chains;
3. Traces of Segre varieties ("Lewy's curves");

all agree. However, one easily constructs examples where they are all different. So when do they agree?

Theorem (Faran, 1981)

If the traces of Segre varieties agree with the chains, then M is locally spherical.

Remark: Faran attributes this question to Lewy. The proof is through a thorough examination of the structure equations; Faran's theorem also holds in $\mathbb{C}^n$ for $n > 2$, where one has to consider intersections of Segre varieties.

A bit newer result

One might wonder about the same question but comparing Lewy's curves to traces of extremal discs (which one probably should refer to as Kobayashi curves).

Theorem (Bertrand-Della Sala-L 2019)

If the traces of Segre varieties agree with the traces of extremal discs, then M is locally spherical.

This actually follows from a bit stronger theorem:

Theorem (Bertrand-Della Sala-L 2019)

If the traces of Segre varieties associated to points on an outward normal at $p \in M$ agree with the traces of extremal discs, then p is an umbilical point.

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And the “newest” result

Theorem (Bertrand-Della Sala-L)

Let $M \subset \mathbb{C}^2$ be strictly pseudoconvex and of class at least $\mathcal{C}^{12}$. If the chains of M are boundaries of stationary discs, then M is locally spherical.

Remarks:

- Again follows from a stronger result (below).
- We need to assume $\mathcal{C}^{12}$ at the moment for reasons to be elaborated.
- For $M \subset \mathbb{C}^n$, where $n > 2$, we speculate an even stronger theorem might be true by current results of Della Sala and coauthors.

On the way to the "real" theorem

Lemma

Let $x_0(.,0), \varphi, \psi, \xi$ be four functions in s of class $\mathcal{C}^7$. Then there exists a family of initial conditions of the form

$$y_1(s,0) = s^2\varphi(s), \quad z_2(s,0) = s + s^5\psi(s), \quad p_{x_0}(s,0) = -\frac{1}{2}s^2 + s^6\chi(s),$$

$$p_{y_1}(s,0) = -\frac{3}{4}, \quad p_{z_2}(s,0) = -\frac{3i}{4}s + s^5\xi(s)$$

for some function χ of class $\mathcal{C}^7$ such that

$$H(x_0(s,0), y_1(s,0), z_2(s,0), p_{x_0}(s,0), p_{y_1}(s,0), p_{z_2}(s,0)) = 0,$$

where H is the Fefferman Hamiltonian associated to ρ.

The real theorem, as promised

Theorem

Let $M \subset (\mathbb{C}^2, 0)$ be a strictly pseudoconvex hypersurface of class $\mathcal{C}^{12}$ with local defining equation of the form

$$\rho = 2\,\mathrm{Re}\, z_1 - \left(|z_2|^2 + 2a|z_2|^4\,\mathrm{Re}\, z_2^2 + (\mathrm{Im}\, z_1) \cdot \eta(\mathrm{Im}\, z_1, z_2, \bar{z}_2) + \delta(z_2, \bar{z}_2) \right),$$

where η and δ are of weighted order $O(6)$ and $O(7)$ respectively. If every chain for M for a family of starting conditions as in the preceding Lemma is the boundary of a stationary disc then $a = 0$.

Weighted expansion of the Hamiltonian

We use a natural grading in order to understand the Hamiltonian we're interested in as a perturbation of the Hamiltonian H_0 associated to

$$\rho_0 = 2\operatorname{Re} z_1 - |z_2|^2 - 2a|z_2|^4 \operatorname{Re} z_2^2$$

which looks like

$$\begin{aligned} H_0 = & 24a(\operatorname{Re} z_2^2) p_{x_0}^2 + \left(2 - \frac{64a}{3}\left(|z_2|^2 \operatorname{Re} z_2^2\right)\right) p_{x_0} p_{y_1} + \frac{ia}{3}\left(24 z_2^2 \bar{z}_2 + 8\bar{z}_2^3\right) p_{x_0} p_{z_2} \\ & - \frac{ia}{3}\left(8 z_2^3 + 24 z_2 \bar{z}_2^2\right) p_{x_0} p_{\bar{z}_2} + \frac{1}{3}\left(-|z_2|^2 + 4a|z_2|^4 \operatorname{Re} z_2^2\right) p_{y_1}^2 \\ & + \frac{i}{3}\left(\bar{z}_2 - a z_2\left(4|z_2|^4 + 6\bar{z}_2^4\right)\right) p_{y_1} p_{z_2} - \frac{i}{3}\left(z_2 - a\bar{z}_2\left(4|z_2|^4 + 6 z_2^4\right)\right) p_{y_1} p_{\bar{z}_2} \\ & - \frac{1}{3}\left(1 - 16a|z_2|^2 \operatorname{Re} z_2^2\right) |p_{z_2}|^2. \end{aligned}$$

It turns out that $H = H_0 + O(7)$ for any perturbation of ϱ_0.

Using the Hamiltonian

If we use the weighted expansion of the Hamiltonian, we can get a number of nice properties of the chains associated to our starting conditions; in particular, we can compute some information of the expansion in s of the family of chains $\gamma(s, t)$ associated to our starting condition, and a good enough control on the period T_s for closed chains $\gamma(s, \cdot)$.
We then take this information and plug it into the stationarity moment conditions, and after some technical detail, get $a = 0$.

Some of the details

We now solve the Hamiltonian system associated to ϱ and get:

Lemma

We have

$$z_2(s,t) = se^{it} - \frac{4}{3}s^5 a e^{3it} + O(s^6), \quad y_1(s,t) = O(s^2).$$

The trick is that this is actually *equivalent* to solving the Hamiltonian system associated to H_0, which...

The Hamiltonian equations for H_0

$$0 = H_0\left(z_2(t), p_{x_0}(t), p_{y_1}(t), p_{z_2}(t)\right), \quad p'_{x_0} = 0, \quad p'_{y_1} = 0$$

$$x'_0 = 48a\left(\operatorname{Re} z_2\right) p_{x_0} + \left(2 - \frac{64a}{3}\left(|z_2|^2 \operatorname{Re} z_2\right)\right) p_{y_1}$$
$$+ \frac{ia}{3}\left(24 z_2^2 \bar{z}_2 + 8\bar{z}_2^3\right) p_{z_2} - \frac{ia}{3}\left(8 z_2^3 + 24 z_2 \bar{z}_2^2\right) p_{\bar{z}_2}$$

$$y'_1 = \left(2 - \frac{64a}{3}(|z_2|^2 \operatorname{Re} z_2^2)\right) p_{x_0} + \frac{2}{3}\left(-|z_2|^2 + 2a|z_2|^4 \operatorname{Re} z_2^2\right) p_{y_1}$$
$$+ \frac{i}{3}\left(\bar{z}_2 - a z_2(4|z_2|^4 + 6\bar{z}_2^4)\right) p_{z_2} - \frac{i}{3}\left(z_2 + a\bar{z}_2(4|z_2|^4 + 6 z_2^4)\right) p_{\bar{z}_2}$$

$$z'_2 = -\frac{16ia}{3}\left(z_2^3 + 3|z_2|^2 \bar{z}_2\right) p_{x_0} - \frac{2i}{3}\left(z_2 - a|z|^2(6z_2^3 - 4 z_2 \bar{z}_2^2)\right) p_{y_1}$$
$$- \frac{2}{3}\left(1 - 16a|z_2|^2 \operatorname{Re} z_2^2\right) p_{z_2}$$

$$p'_{z_2} = -48a\bar{z}_2 p_{x_0}^2 + \frac{64a}{3}\left(z_2^3 + 3 z_2 \bar{z}_2^2\right) p_{x_0} p_{y_1} - \frac{96ai}{3}\left(\operatorname{Re} z_2^2\right) p_{x_0} p_{z_2}$$
$$+ \frac{96ai}{3}|z_2|^2 p_{x_0} p_{\bar{z}_2} + \frac{2}{3}\left(z_2 - 4a|z_2|^2(z_2^3 + 2 z_2 \bar{z}_2^2)\right) p_{y_1}^2 +$$
$$+ \frac{i}{3}\left(-2 + 16a|z_2|^2(z_2^2 + 3\bar{z}_2^2)\right) p_{y_1} p_{z_2} - \frac{12ai}{3}\left(z_2^4 + 2 z_2^2 \bar{z}_2^2\right) p_{y_1} p_{\bar{z}_2}$$

Enforcing stationarity

Now assume that $(z_1(s,\cdot), z_2(s,\cdot))$ coincide with the boundary of a stationary disc $f_s = (g_s, h_s)$. Now h_s is necessarily the Riemann map associated to the domain bounded by the curve S_s parameterized by $z_2(s,\cdot)$.
If f_s is stationary then there exists a continuous function $a_s : b\Delta \to \mathbb{R}^+$ and functions $\widetilde{g}_s, \widetilde{h}_s \in \mathcal{O}(\Delta) \cap C(\overline{\Delta})$ satisfying

$$\widetilde{g}_s(\zeta) = \zeta a_s(\zeta) \frac{\partial \rho}{\partial z_1} \left(g_s(\zeta), h_s(\zeta), \overline{g_s(\zeta)}, \overline{h_s(\zeta)} \right)$$
$$\widetilde{h}_s(\zeta) = \zeta a_s(\zeta) \frac{\partial \rho}{\partial z_2} \left(g_s(\zeta), h_s(\zeta), \overline{g_s(\zeta)}, \overline{h_s(\zeta)} \right)$$

for all $\zeta \in b\Delta$. Normalizing this by setting $\zeta = h_s^{-1}(z)$ we get...

Stationarity condition on S_s

$$G_s(z) = zb_s(z)\frac{\partial\rho}{\partial z_1}\left(w_s(z), z, \overline{w_s(z)}, \bar{z}\right)$$

$$H_s(z) = zb_s(z)\frac{\partial\rho}{\partial z_2}\left(w_s(z), z, \overline{w_s(z)}, \bar{z}\right)$$

and rescale using $\Lambda_s(z) = \frac{z}{s}$ to $\tilde{S}_s$.

The point of using S_s and $\tilde{S}_s$ instead of the unit circle is that we can now plug the approximate formulas for z_2 into the moment conditions

$$\int_{\widetilde{S}_s} z^m G_s(sz)dz = \int_{\widetilde{S}_s} z^m H_s(sz)dz = 0, \quad m \geq 0.$$

These equations can now be analyzed with some care to yield $a = 0$.

Thank you for your attention!