On the validity of the p-Poincaré inequality

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$$\left(\int_{\mathbb{R}^n} |f|^p \, dV \right)^{1/p}$$

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$$\Rightarrow (\dagger) \text{ holds if and only if } \lambda_{1,p}(\Omega) > 0.$$

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\begin{aligned}
f(x) = \int_a^x f'(t)\,dt &\Rightarrow |f(x)| \le \int_a^x |f'(t)|\,dt \\
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Hence, there exists a constant $C > 0$ such that

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(ii) Similarly, if $\Omega \subset \mathbb{R}^n$ is open and bounded in one direction, then $\lambda_{1,p}(\Omega) > 0$.

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and consider for $s > 0$ the bijection

$$\begin{aligned} \mathcal{C}_c^\infty(\Omega) &\longrightarrow \mathcal{C}_c^\infty(s\Omega) \\ f &\mapsto f_s(x) := f(x/s). \end{aligned}$$

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i.e., if (†) holds, then the inradius is finite.

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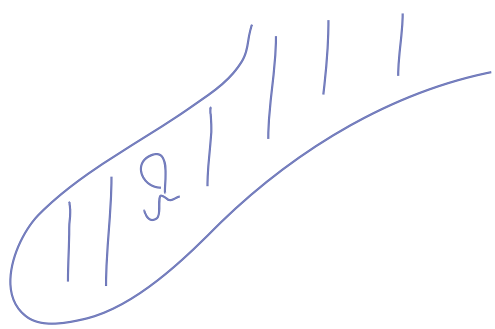

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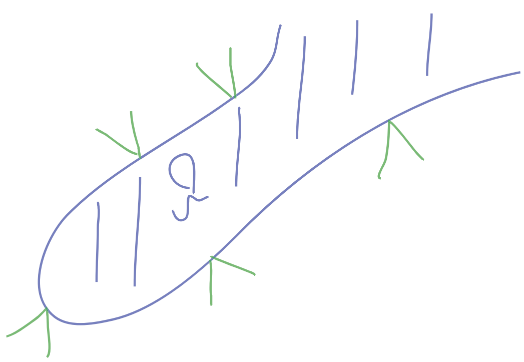

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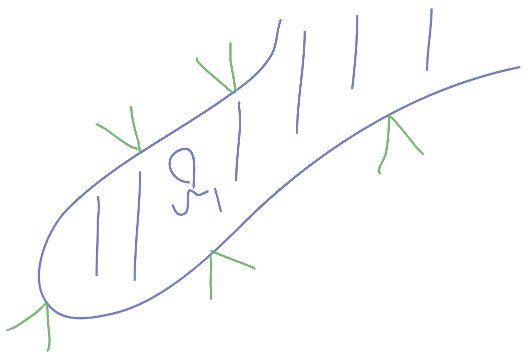

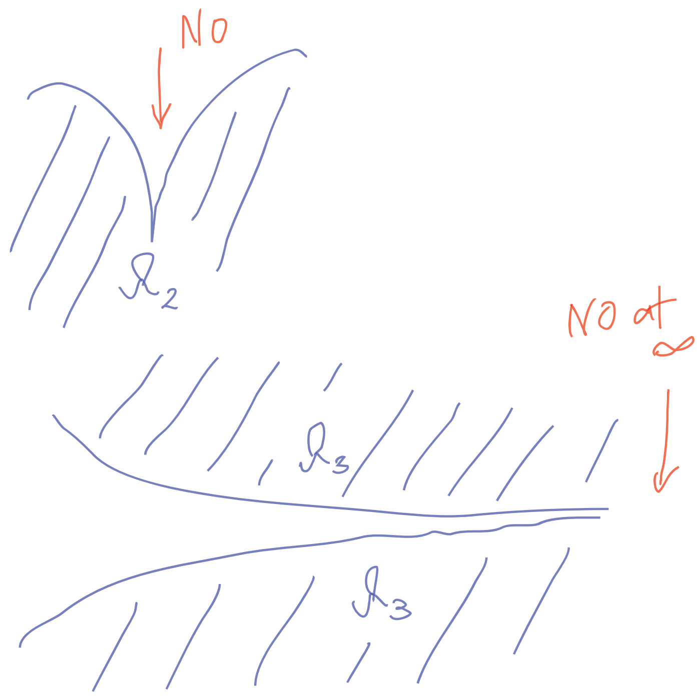

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Set $\varphi_j = (1-u_j)\varphi$

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However, $\mathcal{R}_{\mathbb{R}^n} = \infty$ and $\mathcal{R}_{\mathbb{R}^n \setminus \mathbb{Z}^n} < \infty$.

Capacitary inradius

Definition

Let $K \subset \mathbb{R}^n$ be a compact set. The Sobolev p-capacity of K is

$$C_p(K) = \inf\{\|u\|_p^p + \|\nabla u\|_p^p : u \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ u = 1 \text{ near } K\}.$$

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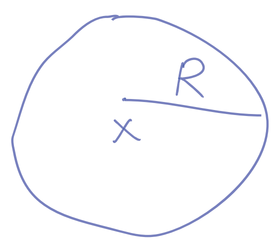

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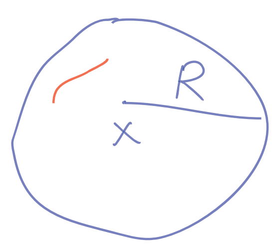
$\forall x$



$K \subset \Omega^c$

$C_p(K) > \epsilon$

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Morrey's : $\|u\|_{\sup} \leq C(n,p) \|u\|_{1,p}$

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- a precursor of this notion was used in Maz'ya–Shubin '05: $p = 2$, $n \geq 3$, Wiener capacity

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  - in G. '23: $n \geq 3$, $p = 2$, Newton capacity
- a precursor of this notion was used in Maz'ya–Shubin '05: $p = 2$, $n \geq 3$, Wiener capacity

$r_\gamma(\Omega)$, $\gamma \in (0,1)$

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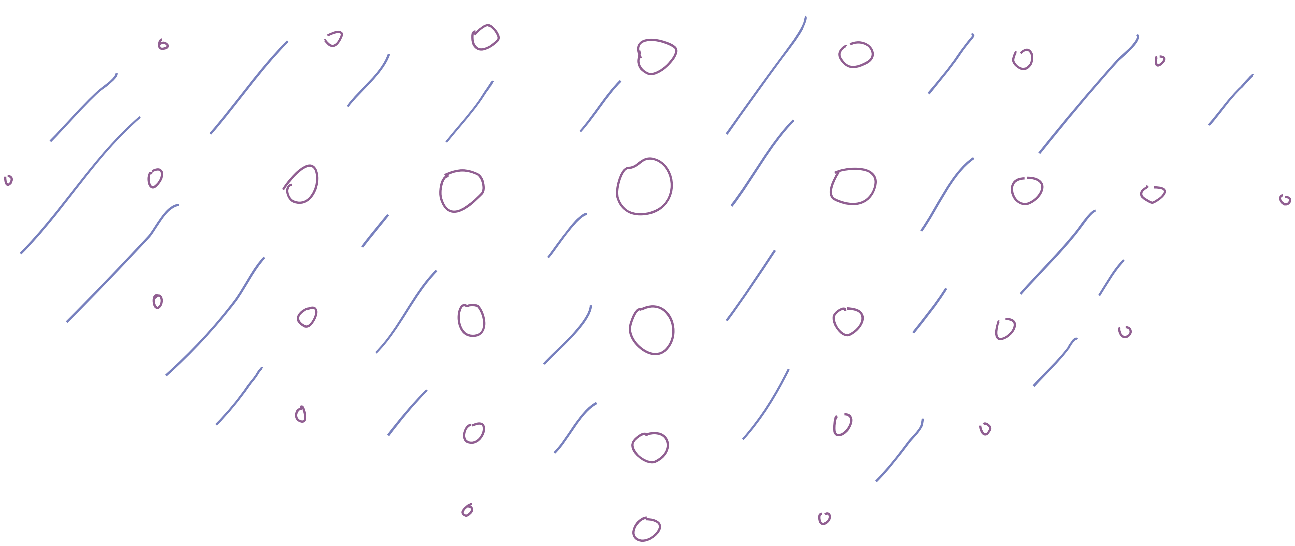

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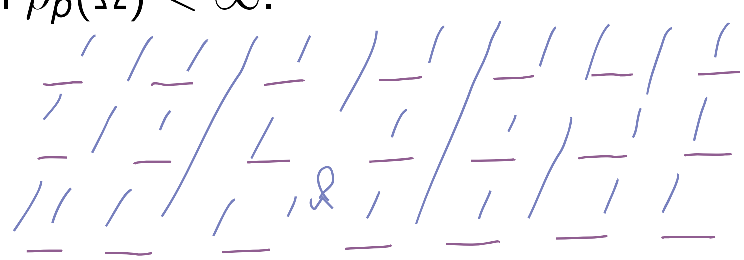

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Let $\Omega \subset \mathbb{R}^n$ *be open,* $1 < p < \infty$*. Then*

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Corollary (G. '24)

Let $K \subset \mathbb{C}^n$ be a compact set. Suppose condition ($\widetilde{P}_n$) holds on K, then condition (P_n) holds on K.

$(P_n)/(\widetilde{P}_n)$

Definition

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$$\|u\| \leq \frac{\sqrt{2}}{\sqrt{M}} \|\bar{\partial}_{n-1}^{U_M} u\| \quad \text{for all } u \in \text{Dom}(\bar{\partial}_{n-1}^{U_M}) \cap (\ker(\bar{\partial}_{n-1}^{U_M}))^\perp$$

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$\lambda_{1,2}(U_M) \geqslant 2M$

$\Rightarrow \delta_2(U_M) \lesssim \frac{1}{\sqrt{M}}$

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(c) *If Ω is connected, unbounded and such that $\rho_p(\Omega) = \rho_p(\Omega \cap \mathbb{B}_R(0))$ for some $R > 0$, then equality cannot hold.*

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$$\frac{\delta_R(\Omega)}{R^n \|E_R\|_{op}} \leq \lambda_{1,p}(\Omega).$$

Here, E_R is a bounded, linear extension operator from $W^{1,p}((0,R)^n)$ to $W^{1,p}((0,R)^n)$ such that $(E_R f)_{|_{(0,R)^n}} = f$.