Complex Dynamics and Complex Geometry

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Abstract

We will discuss dynamics in complex dimension 2. We show some connections between the complex Hénon family and the complex geometry of the currents they generate. Then we look at the parallels with automorphisms of compact surfaces.
The moral of this talk will be that in the process of doing complex dynamics we encounter a number of objects that are "old friends" from complex analysis.
And we will see that in many cases, dynamical behavior and geometry are reflected in each other.

1986 Hubbard approach: complex Hénon maps

View them as small perturbations of the 1-D case

$$H(z, w) = (z^2 + c - aw, z) : \mathbb{C} \to \mathbb{C},$$

$$H^{-1}(z, w) = (w, (w^2 + c - z)/a)$$

$$K^{\pm} = \{(z, w) \in \mathbb{C}^2 : H^{\pm n}(z, w) \not\to \infty, \quad K = K^+ \cap K^-$$

$$J^{\pm} = \partial K^{\pm}, \quad J = J^+ \cap J^-$$

The forward/backward escape loci are

$$U^{\pm} := \mathbb{C}^2 - J^{\pm}$$

As in dimension 1, we define the (forward/backward) Fatou sets $\mathcal{F}^{\pm}$ as the sets of normality for the families $\{H^{\pm n} : n \geq 0\}$, and the Julia sets $J^{\pm}$ are defined as the complements of the Fatou sets. The following is immediate:

Theorem

The (forward/backward) Fatou sets $\mathcal{F}^{\pm} = int(K)^{\pm} \cup U^{\pm}$.

The Green function for the forward escape locus may be defined using the same dynamical formula as in dimension 1.

Theorem (Hubbard)

The limit

$$G^+ := \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|H^n(z,w)\|$$

converges uniformly on $\mathbb{C}^2$ *and satifies*

- $G^+ \circ H = 2G^+$
- $U^+ := \mathbb{C}^2 - K^+ = \{G^+ > 0\}$

The Green function is pluriharmonic on U^+, and ∂G^+ induces a holomorphic foliation $\mathcal{G}^+$ on U^+. Hubbard's first approach was to study the properties of U^+ and $\mathcal{G}^+$.

Note: K^+ is never compact.

Why is this interesting for SCV?

These are like "rough versions" of objects we know well.

G^+ is pluri-harmonic on $\{G^+ > 0\}$ and $\partial G^+ \neq 0$

$$J^+ = \lim_{\epsilon \to 0} \{G^+ = \epsilon\}$$

so *J^+ is a (generalized) "Levi flat hypersurface"*

In principle, J^+ is laminated by Riemann surfaces, and its cross section is like a 1-dimensional Julia set.

Generically, $\partial G^+ \wedge \partial G^- \neq 0$, and

$$J = \lim_{\epsilon \to 0} \{G^+ = G^- = \epsilon\}$$

so $J = J^+ \cap J^-$ is "totally real".

Theorem (Hubbard-ObersteVorth)

The topology of the set U_H^+ is independent of H

$K^{\pm}$ and $U^{\pm}$ are very interesting from the point of Complex Analysis/Geometry.
Hubbard conjectured and gave evidence for: *If H_1 and H_2 are two Hénon maps, and if $U_{H_1}^+$ is biholomorphically equivalent to $U_{H_2}^+$, then $H_1 = H_2$.*

Given a map $g : S \to S$, an invertible map is obtained from the *projective limit*, which is the shift map acting on the sequence space

$$\{(s_n)_{n\in\mathbb{Z}} : s_{n+1} = g(s_n)\} \subset S^{\mathbb{Z}}$$

A validation of the Hubbard approach was given by:

Theorem (Fornæss-Sibony, Hubbard-ObersteVorth)

If $p(z) = z^2 + c$ is expanding on its Julia set J_p, and if $|a|$ is sufficiently small, then the restriction $H|_J : J \to J$ is topologically conjugate to the projective limit of the 1-dimensional map $p|_{J_p} : J_p \to J_p$.

Approach #2: consider *all* polynomial automorphisms of $\mathbb{C}^2$

Use the structure of the group – Jung's Theorem

Theorem (Friedland-Milnor)

If f is a polynomial automorphism of $\mathbb{C}^2$, then either

- *f has elementary dynamics.*
- *f is conjugate to a map of the form $f_1 \circ \cdots \circ f_N$, where $f_j(x, y) = (p_j(x) - \delta_j y, x)$, p_j polynomial and $\delta_j \in \mathbb{C}$, $\delta_j \neq 0$*

Other results in the F-M paper show that if you study the map H, then you are actually studying the nontrivial polynomial automorphisms of $\mathbb{C}^2$. In a similar way, we define G^+ and $U^\pm$, etc., for $f = f_1 \circ \cdots \circ f_N$.

One roadblock to studying polynomial automorphisms of $\mathbb{C}^3$ is that there is no known analogue of Jung's Theorem, which gives a convenient set of generators for the automorphism group. *If you wanted to start working in $\mathbb{C}^3$, what maps would you study first?*

Is there a *special* map in the Hénon family?

In dimension 1, there are the maps $z \mapsto z^d$ for $d \in \mathbb{Z}$, as well as the Chebyshev polynomials T_d. These maps are special in many ways, including the fact that their Julia sets are "familiar" sets – the circle and the interval. These models get special attention.

However, for Hénon maps, it seems that there is nothing analogous.

Theorem (B-Kyounghee Kim)

If f is a Hénon map, then the forward/backward Julia sets $J^{\pm}$ are never

- *C^1 smooth, as a manifold-with-boundary*
- *Semi-analytic.*

Revisit Hubbard's conjecture about the escape locus

Maps are determined by their dynamical sets

The topological type of the escape locus U_f depends only on the degree of $f = f_1 \circ \cdots \circ f_N$.

Hubbard's conjecture has developed into a very striking general result:

Theorem (Ratna Pal)

Suppose that $f = f_1 \circ \cdots \circ f_N$ and $f' = f'_1 \circ \cdots \circ f'_{N'}$ are Hénon maps. If the escape locus U_f^+ is biholomorphically equivalent to $U_{f'}^+$, then f and f' are (essentially) the same.

Meaning of "essentialy":

- f and $f_2 \circ \cdots \circ f_N \circ f_1$ are conjugate, so have the "same" escape locus.
- We may rotate f into f' by certain $d^2(d-1)$-th roots of unity.

Approach #3: the sets $K^{\pm}$ and the currents on them

The response of B and Sibony upon seeing Hubbard's talks:

"What can we say about $\mu^{\pm}$ and μ?"

We define:

$$\mu^{\pm} := dd^c G^{\pm} = 2i\partial\bar{\partial}G^{\pm}$$

It follows easily that

$$\text{support}(\mu^{\pm}) = J^{\pm}$$

If d is the degree of f, we have $G^{\pm} \circ f = d^{\pm 1} G^{\pm}$, which gives

$$f^*\mu^+ = d \cdot \mu^+ \text{ and } f^*\mu^- = d^{-1}\mu^-.$$

If $L \subset \mathbb{C}^2$, the *slice measures* are given by $\mu^+|L := dd^c G^+|_L$. This is the classical harmonic measure for the set $K^+ \cap L$ inside $L \cong \mathbb{C}$. Thus we think of μ^+ as *the universal harmonic measure* for K^+.

We may define the measure $\mu := \mu^+ \wedge \mu^-$, and it follows that

$$f^*\mu = f_*\mu = \mu$$

Interlude on pluri-potential theory

If $E \subset \mathbb{C}^n$ is compact, then the *pluri-complex Green function* is given by

$$U_E(z) = \sup\{v(z) : v \leq 0 \text{ on } E, \text{ and } v \text{ grows no faster than logarithm}\}$$

The *equilibrium measure* is given by the Monge-Ampère operator

$$\mu_E := (dd^c U_E)^n.$$

Theorem (B-Taylor)

The support of μ_E is the Shilov boundary of E.

We apply this to our mapping f, and we obtain:

Theorem (B-Sibony)

The Green function of K is $U_K = \max\{G^+, G^-\}$, and $\mu = \mu_K = \mu^+ \wedge \mu^-$.

How many Julia sets are there?

$J = J^*$?

We have the "classical" Julia sets $J^{\pm}$, where the forward or backward iterates do not make a normal family.
We also have the set $J := J^+ \cap J^-$.
We may also set

$$J^* := \text{Shilov boundary}(K) = \text{support}(\mu)$$

Clearly, we always have $J^* \subset J$.

Open Question: Is it always the case that $J = J^*$?

This question has been intensively studied and may be one of the hardest in the subject.

Attraction and basins

Given a closed set E, we set

$$W^s(E) = \{(z, w) \in \mathbb{C}^2 : \mathrm{dist}(f^n(z, w), E) \to 0\}$$

Theorem

If E is an attracting fixed (periodic) point, then $W^s(E) \cong \mathbb{C}^2$.
If E is a saddle (periodic) point, then $W^s(E) \cong \mathbb{C}$.

Basins of attracting fixed points:

- The original Fatou Bieberbach domain was obtained by taking a cubic Hénon map $f(x, y) = (x^3 - ax - by, x)$ with suitable a, b to have attracting fixed points $\pm P$. In this case there are two basins, $W^s(P)$ and $W^s(-P)$, which must be disjoint and are both $\cong \mathbb{C}^2$.
- In 1974, Newhouse showed that there are real Hénon maps f_R for which there are infinitely many (real) attracting cycles p_j. For each of these, the (real) basin, $W^s(p_j, f_R) \subset K \cap \mathbb{R}^2$ is an (open) disk. Each basin $W^s(p_j, f_R)$ in $\mathbb{C}^2$ is a Fatou-Bieberbach domain, and of course intersects $\mathbb{R}^2$ in a disk.

Two approaches to dynamics

Orbits of points

Given a dynamical system $f : X \to X$, we may study *point orbits*: the sets

$$\mathcal{O}(x) := \{f^n(x) : n \in \mathbb{Z}\}$$

What happens to a point $x \in X$ as we iterate it?

Action on 1-dimensional sets

- Suppose that $E \subset \mathbb{C}^2$ is an algebraic curve or analytic disk, and consider the map

  $$E \mapsto f(E) \text{ or } E \mapsto f^{-1}$$

  How does $f^n(E)$ behave as $n \to \pm\infty$?
- Let $\mathcal{T}$ denote the set of positive, closed (1,1)-currents.
  What is the behavior of $f^* : \mathcal{T} \to \mathcal{T}$?

Pass to the compactification $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$

One of my favorite theorems is by Fornæss and Sibony

If a current has a local potential, i.e., $T = dd^c u$ locally, then we pull it back by setting $f^*T := dd^c(u \circ f)$.

Theorem

The operator $dd^c = 2i\partial\bar{\partial}$ sets up an isomorphism between the Lelong class

$$\mathcal{L} = \{psh\ u\ on\ \mathbb{C}^2 : u(z) \leq \log^+ \|z\| + O(1)\}$$

and

$$\mathcal{T}_1 = \{positive,\ closed\ currents\ on\ \mathbb{P}^2,\ \ no\ mass\ at\ L_\infty,\ \ total\ mass\ 1\}$$

A Hénon map f extends to a birational map of $\mathbb{P}^2$, and $\frac{1}{d}f^* : \mathcal{T}_1 \to \mathcal{T}_1$.

Theorem (Fornæss-Sibony)

- *The current μ^+ is the unique global attractor for $\frac{1}{d}f^*$ acting on $\mathcal{T}_1$*
- *If T is pos, closed on $\mathbb{C}^2$, and if Supp$(T) \subset K^+$, then $T = \mu^+$.*

Advantages of studying μ^+ instead of J^+

Using currents leads to a different sense of convergence

Let $M \subset \mathbb{R}^n$ be an oriented k-manifold. The *current of integration* $[M]$ is defined by its action on the space of test k-forms:

$$\langle [M], \varphi \rangle := \int_M \varphi$$

Passage manifold $\rightsquigarrow$ current of integration behaves well under mapping: $f^*[M] = [f^{-1}M]$ and $f_*[M] = [f(M)]$. And $d[M] = [\partial M]$. *Convergence of currents* is defined as

$$\lim_{j\to\infty} T_j = T \text{ if } \forall \varphi,\ \langle T_j, \varphi \rangle \to \langle T, \varphi \rangle.$$

Note that $[D^s]$ is not closed since $d[D^s] = [\partial D^s]$, but we have:

Theorem (B-Smillie)

Let $\mathcal{D} \subset \mathbb{C}^2$ be a complex disk, and let $c =$ Mass of slice measure $\mu^-|_{\mathcal{D}}$. Then

$$\frac{1}{d^n}[\mathcal{D}] = \frac{1}{d^n}[f^{-n}\mathcal{D}] \to c\mu^+$$

Advantages of studying μ^+ instead of J^+

Everything seems to "happen" at J^+

Corollary

If p is a saddle point, then $W^s(p) \cong \mathbb{C}$ is a dense subset of J^+.
In particular, if p_1 and p_2 are saddle points, then $\overline{W^s(p_1)} = \overline{W^s(p_2)}$.

Proof: Let $p \in D^s \subset W^s(p)$ be a complex disk. Then $\mu^-|_{D^s} \neq 0$. Further, $D^s \subset f^{-1}(D^s) \subset \cdots$, so $W^s(p) = \bigcup f^{-n}(D^s)$ dense in $\mathrm{Supp}(\mu^-)$.

Corollary

Let p be an attracting fixed point, and let $\mathcal{B}(p) \cong \mathbb{C}^2$ be its basin of attraction. Then $\partial\mathcal{B}(p) = J^+$.
In particular if $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ are basins, then they all have the same boundary.

Proof: Similar argument.

Thus $W^s(p)$'s and $\mathcal{B}(p)$'s must be strangely imbedded and intertwined.

Structure of μ^+

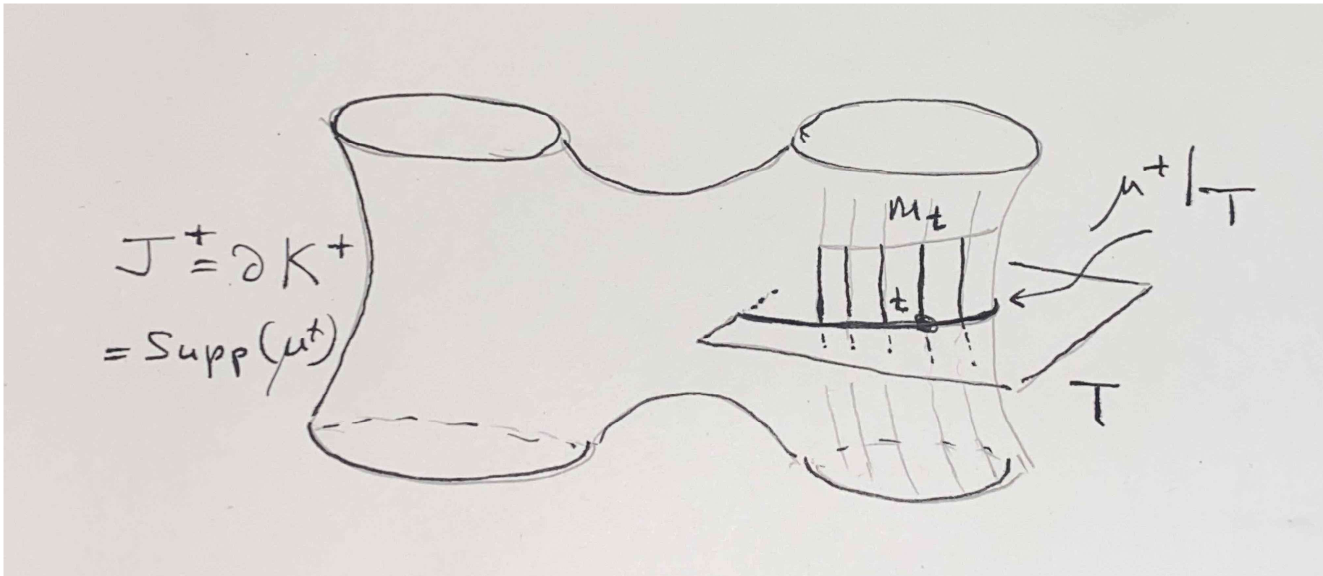
Or, more accurately, the picture that is in my mind

J^+ has a lamination $\mathcal{W}^s$ by stable manifolds; T is a transversal, and the slice measures serve to *"measure"* the lamination $\mathcal{W}^s$.

μ^+ is a *geometric current* made out of slice measures $\mu^+|_T$ and currents of integration over pieces of stable manifolds M_t, $t \in T$.

This means that, locally, $\mu^+ = \int_{t\in T}[M_t]d(\mu^+|_T)$, acting on test form φ:

$$\langle \mu^+, \varphi \rangle := \int_{t\in T} \langle [M_t], \varphi \rangle \; d(\mu^+|_T) = \int_{t\in T} \int_{M_t} \varphi \; d(\mu^+|_T)$$



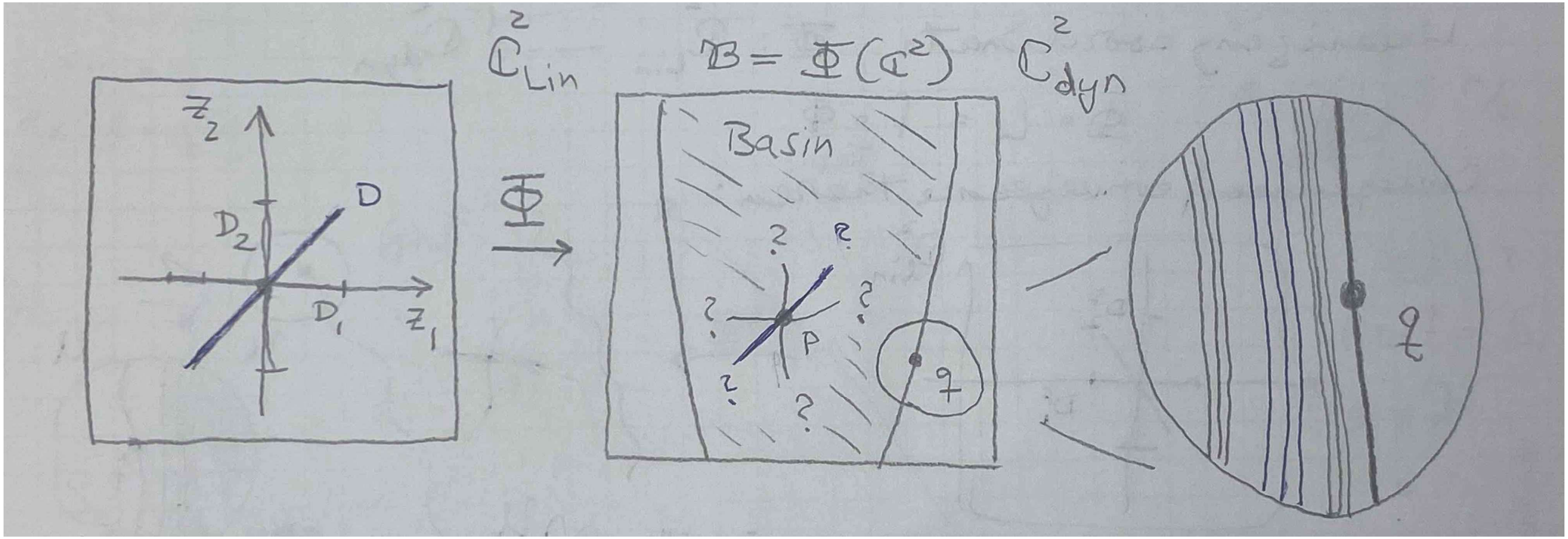
Distortion of the Fatou-Bieberbach imbedding

Local linearization of map at attracting fixed point extends to global imbedding of $\mathbb{C}^2$

$f(p) = p$, $Df_p = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = L$, $|a| < 1$, $\exists \Phi : \mathbb{C}^2_{\mathrm{Lin}} \to \mathbb{C}^2_{\mathrm{dyn}}$, $\Phi \circ L = f \circ \Phi$

Consequences of convergence of the currents $d^{-n}[f^{-n}D]$

- *non-vanishing mass* $\Rightarrow$ $\mathrm{Area}(f^{-n}(\Phi(D)) \cap U) \sim cd^n$
- $\mathrm{Area}(\Phi(D(r)) \cap U) \sim cr^{\mu}$, with $\mu = \log(d)/\log(|a|^{-1}) > 0$
- *convergence of currents* $\Rightarrow$ components of $\Phi(f^{-n}D \cap U)$ must "line up" with the lamination of $\partial\mathcal{B} = J^+$.



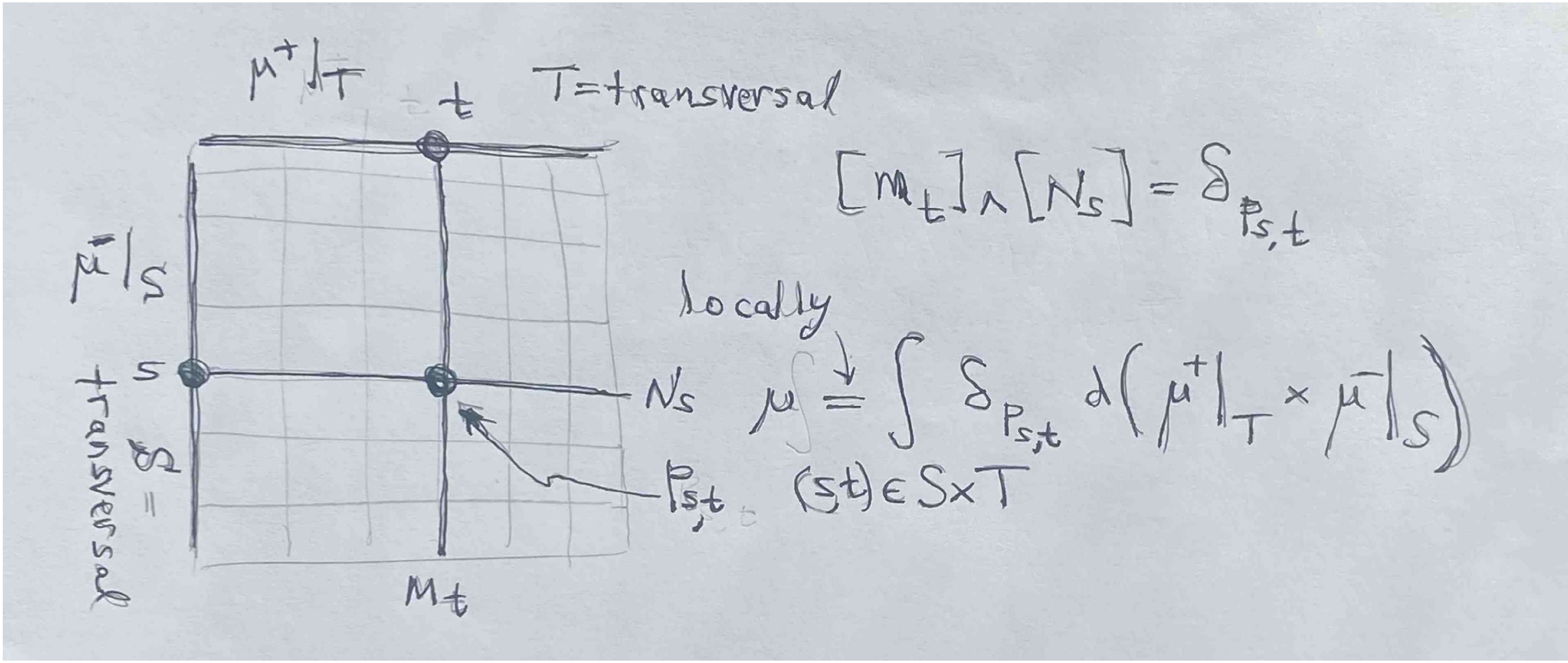
Structure of $\mu = \mu^+ \wedge \mu^-$

Monge-Ampère can also be interpreted as a geometric intersection product.
This is crucial for further dynamical applications.

Analytic definition of the measure $\mu = \mu^+ \wedge \mu^-$:

$$\int \varphi \, \mu := \int G^+ \, dd^c \varphi \wedge \mu^-$$

Geometric intersection:



Ergodic theory consequences of product structure

μ gives distribution of saddle points and J^* gives heteroclinic classes

Theorem (B-Lyubich-Smillie)

Let f be a complex Hénon map, and let $\mathcal{S}_n$ be the saddle cycles of period n

- *For $p_1, p_2 \in \mathcal{S}$, $\overline{W^s(p_1) \cap W^u(p_2)} = J^*$*
- $\overline{\mathcal{S}} = J^*$
- $\mu = \lim_{n\to\infty} \frac{1}{d^n} \sum_{p \in \mathcal{S}_n} \delta_p$

Interlude: Simplest map in dimension 2
Product map, an Endomorphism of $\mathbb{P}^2$

Apply previous approach to a very different map: high vs low topological degree.
$\deg_{\mathrm{alg}}(F) = 2 < \deg_{\mathrm{top}}(F) = 4$

$$F(z,w) = (z^2, w^2) : \mathbb{C}^2 \to \mathbb{C}^2$$

$$[z : w : \xi] \mapsto [z^2 : w^2 : \xi^2] : \mathbb{P}^2 \to \mathbb{P}^2$$

$$G(z,w) := \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|F^n(z,w)\| = \max\{\log|z|, \log|w|, 0\}$$

Currents:

$$T = dd^c G, \quad T^2 := T \wedge T = \mu = d\theta \wedge d\phi|_{\{|z|=|w|=1\}}$$

$$F^* T = 2T, \quad F^*(T^2) = 4T^2$$

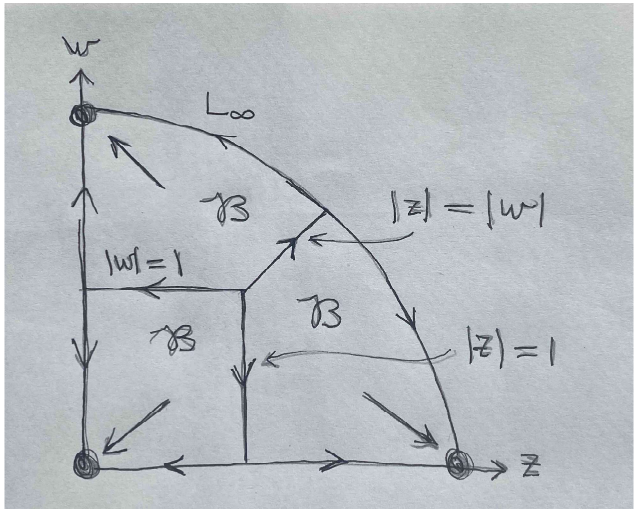
Julia sets:

$$J_1 := \mathrm{Supp}(T), \quad J_2 := \mathrm{Supp}(T^2)$$

Fatou Set for $F(z, w) = (z^2, w^2)$

The Fatou set, i.e., where the iterates are a normal family, is

$$\mathcal{F} = \text{three attracting basins} = \bigcup_{j=1}^{3} \mathcal{B}_j = \mathbb{P}^2 - J_1$$

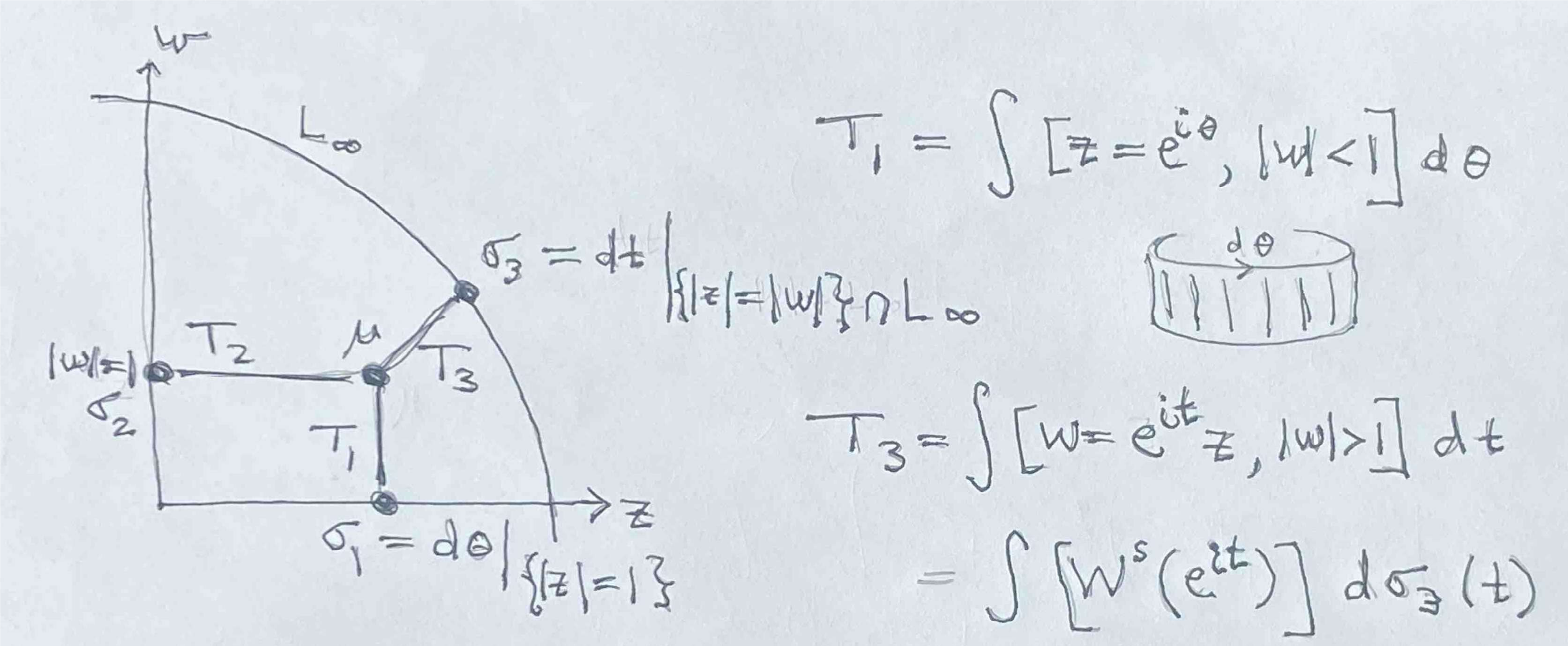


What about J_2? The definition of the Monge-Ampère operator $(dd^c)^2$ has a rather indirect analytical definition (integration by parts).

When we are working with a dynamically generated potential, such as the G above, does $(dd^c)^2$ have definition that “makes sense” dynamically or geometrically?

In the Hénon case, it was technically important that we could show that it was given by a geometric intersection product.

Laminar structure of $T = dd^c G = \max\{\log|z|, \log|w|, 0\}$



F is hyperbolic; μ is index 2 (repelling), and σ_j, $j = 1, 2, 3$ are index 1 (saddle type). F is structurally stable, so μ and σ_j and the laminar structure all persist under small perturbation.

Can we write $\mu = (T_1 + T_2 + T_3)^2$ as a geometric intersection product? Can this geometric intersection be robust enough to work for small perturbations of F?

We return to invertible maps in dimension 2.

We will be led to rational surface automorphisms

Theorem (Cantat)

If X is a compact, complex 2-manifold, and $F : X \to X$ is a holomorphic automorphism with "nontrivial dynamics", then X is one of the following:

- *a torus $\mathbb{C}^2/\mathcal{L}$ and F linear,*
- *a $K3$ surface or a quotient of one, or*
- *a rational surface (to be explained later).*

The parameter spaces of the first two cases (torus and $K3$) are of small dimension, whereas there are spaces of rational automorphisms of arbitrarily large dimension. So we consider this case to be the "principal" one.

A *rational map* $F = [F_0 : F_1 : F_2] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by a triple of homogeneous polynomials of the same degree d, with no common factor. d is the *degree* of F. If F has an inverse, it is *birational*. d is not a birational invariant, but the *dynamical degree* is

$$\mathrm{ddeg}(F) := \lim_{n\to\infty} (\deg(F^n))$$

Note that our original map $H(x, y) = (x^2 + c - ay, x)$ can be written in rational form as $H[x : y : z] = [x^2 + cz^2 - ayz : xz : z^2]$. The inverse is given by $H^{-1}[x : y : z] = [yz : (y^2 + cz^2 - xz)/a : z^2]$.

$$(\text{Note that } H \circ H^{-1} = z^3\,[x : y : z].)$$

For any Hénon map f, its dynamical degree is the same as its degree. Thus $\mathrm{ddeg}(f)$ is an integer. On the other hand,

Theorem

If $F : X \to X$ is a compact surface automorphism, and if $\mathrm{ddeg}(F) > 1$, then $\mathrm{ddeg}(F)$ is algebraic but irrational.

In other words, compact surface automorphisms are never birationally equivalent to Hénon maps.

Invariant currents for compact surface automorphisms

Let $f : X \to X$ be a compact surface automorphism with $\mathrm{ddeg}(f) > 1$, then $f^* : H^2(X) \to H^2(X)$ has a unique eigenvector α^+ such that $f^*\alpha^+ = \mathrm{ddeg}\alpha^+$. Thus there exists a smooth γ^+ such that

$$f^*(\alpha^+) = \mathrm{ddeg} \cdot \alpha^+ + dd^c\gamma^+$$

It follows that

$$G^+ := \sum_{n=0}^{\infty} \frac{\gamma^+ \circ f^n}{\mathrm{ddeg}^n}$$

converges to a continuous function on X, and $\mu^+ := \alpha^+ + dd^c G^+$ is invariant, i.e., $f^*\mu^+ = \mathrm{ddeg} \cdot \mu^+$.

Theorem (Cantat)

Let $f : X \to X$ be a compact surface automorphism with $\mathrm{ddeg}(f) > 1$. Then $\mu^\pm$ and μ give dynamical results parallel to the case of Hénon maps.

Examples of rational surface automorphisms

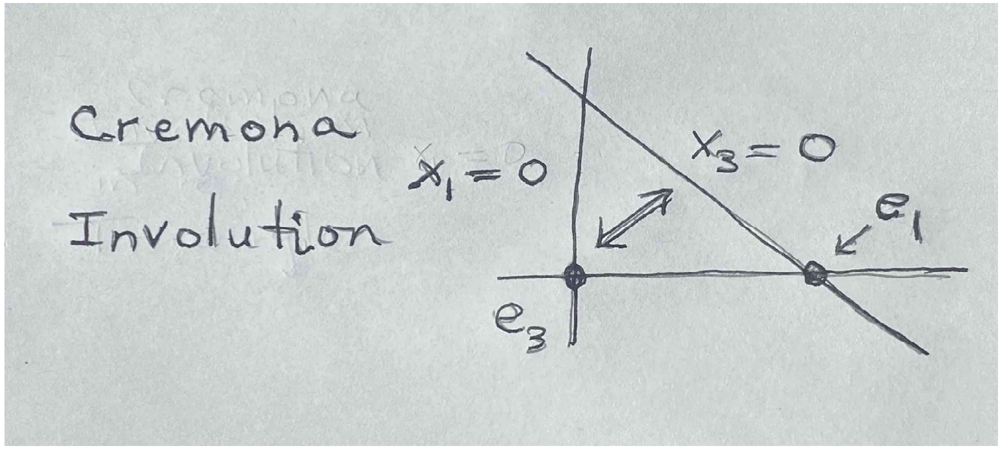
Theorem (Nagata)

If $F : X \to X$ is a rational surface automorphism with ddeg > 1, then there is a birational map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and a blowup $\pi : X \to \mathbb{P}^2$ such that F is the induced map $F = \pi^{-1} \circ g \circ \pi$.

The set of birational maps of the plane is large, and it is an open question which birational maps will lift to automorphisms.

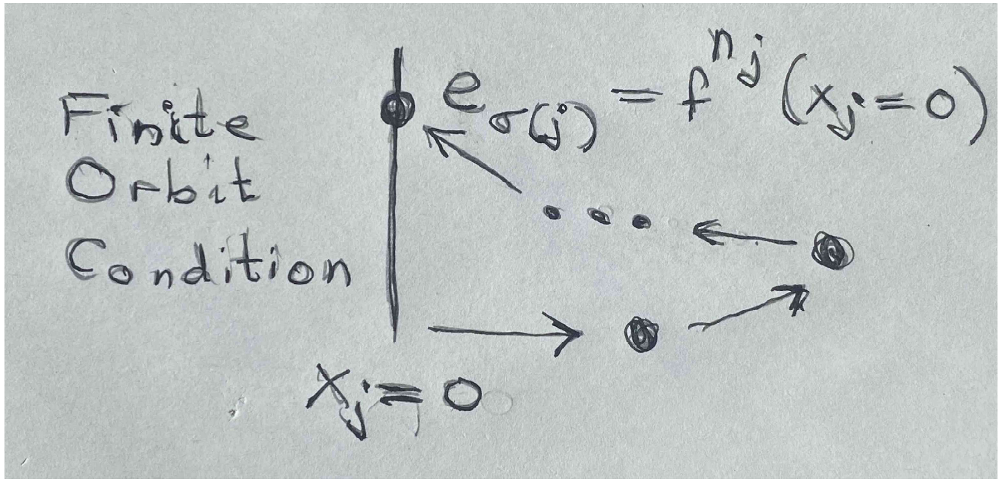
In degree 2, the simplest birational map is the *Cremona involution2*

$$J[x_1 : x_2 : x_3] := \left[\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\right] = [x_2x_3 : x_1x_3 : x_1x_2]$$



The quadratic family $f := L \circ J = L \cdot [yz, xz, xy]^t$

Work with the simplest family: Cremona involution followed by linear, so each coordinate of f is a linear combination of just yz, xz, xy



The Finite Orbit Condition says: *There is a permutation σ of $\{1, 2, 3\}$ and three numbers n_1, n_2, n_3 such that $f^{n_j}(\{x_j = 0\}) = e_{\sigma(j)}$.* This is equivalent to a polynomial condition on the coefficients of L.

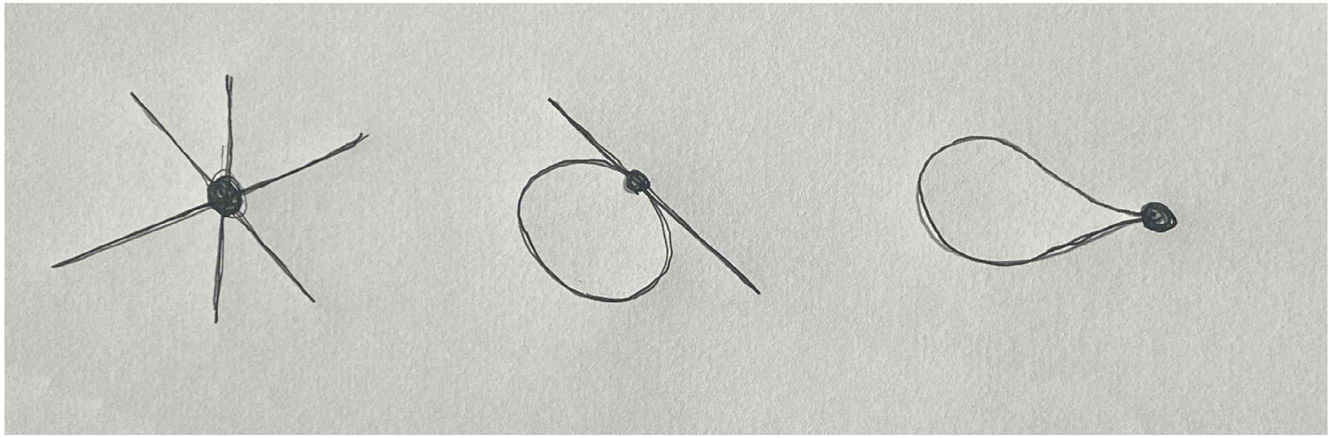
Theorem (B - Kyounghee Kim)

A map of the form $f = L \circ J$ lifts to an automorphism if and only if it satisfies the finite orbit condition.

Theorem (Diller)

Let integers $n_1, n_2, n_3 > 0$ and a permutation σ of $\{1,2,3\}$ be given.* *Then there is a quadratic automorphism realizing the orbit data $((n_1, n_2, n_3), \sigma)$. Further, ddeg > 1 iff $n_1 + n_2 + n_3 \geq 10$.*

- The "$*$" in the statement means that there are some exceptional cases.
- The automorphisms in Diller's Theorem all have invariant curves; the only possibilities are the cubics pictured. With some exceptions, we are able to freely choose the invariant curve.



- This gives a rather simple algebraic construction of the matrix L.

Conjecture: "Most" automorphisms $f := L \circ J$ do not have invariant curves.

Maps with invariant curves are like a generalization of Hénon maps, since the (birational) Hénon map $H : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has the line at infinity L_∞ as an invariant curve.

If $\mathcal{C} = \{q = 0\}$ is an invariant curve for the rational map $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$, then the (meromorphic) 2-form $\eta = dx \wedge dy/q(x, y)$ is invariant: $f^*\eta = \alpha\eta$ for some $\alpha \in \mathbb{C}$.

Theorem (McMullen and . . .)

Let $f = L \circ J : X \to X$ be a quadratic automorphism with an invariant curve $\mathcal{C}$. Then the singular point P of $\mathcal{C}$ is a fixed point for f, and P belongs to the Fatou set of f or f^{-1}. If $ddeg > 1$, then α is a root for the minimal polynomial of $ddeg$. Further, either $|\alpha| = 1$ or $\alpha \in \mathbb{R}$, in which case it is $ddeg$ or $ddeg^{-1}$.

- *If $\alpha = ddeg^{-1} < 1$. Then P is an attracting fixed point. In this case, the Fatou set $\mathcal{F} = \mathcal{B}(P)$ and has full volume in X.*
- *If $|\alpha| = 1$, then there is a Fatou component Ω containing P which is invariant under a torus $\mathbb{T}^1$-action.*

Fatou dynamics

In dimension 1, Fatou classified the periodic F-components. By Sullivan, every F-component is pre-periodic. If $\mathcal{F} \neq \emptyset$, we know the dynamics on a dense, open set.

Question: If $\mathcal{F} \neq \emptyset$, is it necessarily dense in X?

Question: Can we make a classification of periodic Fatou components for surface automorphisms? Maybe something like:

- Attracting basins
  1. $\mathcal{B}$(point) $\cong \mathbb{C}^2$ ✓✓✓
  2. $\mathcal{B}$(rotational disk) $\cong D \times \mathbb{C}$???
  3. $\mathcal{B}$(rotational annulus) $\cong A \times C$???
- Basin of parabolic or semi-parbolic point or disk or annulus ???
- Rotation domains ✓✓✓(Invariant domains Ω such that the iterates $\{f^n|_\Omega\}$ generate a torus $\mathbb{T}^j$, $j = 1, 2$.)
  What are the possibilities for Ω as a domain in $\mathbb{C}^2$?
- Something else ???